EB mixing because of incomplete sky coverage

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Overview of the problem

- Multipoles from (Q, U) data:

$$\underbrace{\begin{pmatrix} \tilde{a}^E_{lm} \\ \tilde{a}^B_{lm} \end{pmatrix}}_{\tilde{\mathbf{a}}_{lm}} = \int d\vec{n} \underbrace{\begin{pmatrix} -X^\dagger_{1,lm} & -iX^\dagger_{2,lm} \\ iX^\dagger_{2,lm} & -X^\dagger_{1,lm} \end{pmatrix}}_{\mathbf{X}^\dagger_{lm}} \underbrace{\begin{pmatrix} Q(\vec{n}) \\ U(\vec{n}) \end{pmatrix}}_{\mathbf{P}}$$

The $\mathbf{X}_{lm}$ is an orthonormal basis on the sphere:
$\int d\vec{n}\mathbf{X}_{lm}(\vec{n})\mathbf{X}^\dagger_{l'm'}(\vec{n}) = \delta_{ll'}\delta_{mm'}\mathbf{I}$

- but incomplete sky coverage:

$$\tilde{\mathbf{a}}_{lm} = \int d\vec{n} W(\vec{n})\mathbf{X}^\dagger_{lm}(\vec{n})\mathbf{P}(\vec{n})$$

The $\mathbf{X}_{lm}$ is no more a basis:
$\int d\vec{n} W(\vec{n})\mathbf{X}_{lm}(\vec{n})\mathbf{X}^\dagger_{l'm'}(\vec{n}) \neq \delta_{ll'}\delta_{mm'}\mathbf{I}$

How the problem can be solved

Estimation of power spectra

- Quadratic estimators of $C_l^{E(B)}$ are biased because of $l-$modes mixing and EB mixing
- As $C_l^B \ll C_l^E$, B power spectra covariance dominated by E contributions
  $\longrightarrow$ big error bars

Solving the mixing

- Find an orthonormal basis on the cut sky (Bunn et al.)
- Remove the contributions of ambiguous modes
  - Use alternative fields pure in E or B mode (Smith & Zaldarriaga)
  - Spin weighted counterterms (Smith)

Polarized fields on incomplete sky

Polarized fields on the sphere

Any polarized fields can be constructed from the sum of *pure* E-modes, Ψ_E, and *pure* B-modes, Ψ_B :

$$\mathbf{P}(\vec{n}) = \mathbf{D}_E \Psi_E(\vec{n}) + \mathbf{D}_B \Psi_B(\vec{n})$$

with the condition on the sphere

$$\mathbf{D}_E^\dagger \mathbf{D}_B = \mathbf{D}_B^\dagger \mathbf{D}_E = 0 \quad \text{and} \quad \mathbf{D}_E^\dagger \mathbf{D}_E = \mathbf{D}_B^\dagger \mathbf{D}_B \simeq \nabla^4$$

Polarized fields on incomplete sky

The E/B decomposition on a part of the sky is not unique: E and B subspaces overlap

$\rightarrow$ Some modes verify the E and B conditions : $\mathbf{D}_E^\dagger \mathbf{P}_a = 0$ et $\mathbf{D}_B^\dagger \mathbf{P}_a = 0$

$\rightarrow$ Polarization field is decomposed into *pure* E modes, *pure* B modes and *ambiguous* modes

Polarized fields on incomplete sky

Flat sky approximation

- **Ambiguous modes** constructed as E modes which satisfies the B mode condition : $\mathbf{P}_a = \mathbf{D}_E \Psi$ with $\mathbf{D}_E^\dagger \mathbf{P}_a = 0$
  $\longrightarrow$ ambiguous modes are the one satisfying $\nabla^4 \Psi = 0$
- **Pure** E **modes** orthogonal to all B modes (pure or ambiguous) : $\int_\Omega \mathbf{P}_E \cdot (\mathbf{D}_B \Psi_B) d\Omega = 0$
  $\longrightarrow$ Ψ_E verify Dirichlet and Neumann boundary conditions
  $\longrightarrow$ To find eigenfunctions of ∇^4 satisfying the boundary requirement and apply the $\mathbf{D}_{E(B)}$ operator to these functions for deriving the $E(B)$ modes.

To summarize

- Ambiguous modes are given by eignefunctions of ∇^4 with vanishing eigenvalue
- *pure* E modes are given by eigenfunctions of ∇^4 which verifies the Dirichlet and Neumann boundary conditions

Filtering the ambiguous modes

Complete sky coverage

- Polarized fields decomposed into E and B modes : $\mathbf{P} = \mathbf{D}_E \Psi_E + \mathbf{D}_B \Psi_B$
- Apply the $\mathbf{D}^\dagger_{E(B)}$ to $\mathbf{P}$: $\mathbf{D}^\dagger_E \mathbf{P} = \mathbf{D}^\dagger_E \mathbf{D}_E \Psi_E$ and $\mathbf{D}^\dagger_B \mathbf{P} = \mathbf{D}^\dagger_B \mathbf{D}_B \Psi_B$
  because $\mathbf{D}^\dagger_E \mathbf{D}_B = \mathbf{D}^\dagger_B \mathbf{D}_E = 0$

Incomplete sky coverage

- Polarized fields decomposed into pure E, pure B and ambiguous modes : $\mathbf{P} = \mathbf{D}_E \Psi_E + \mathbf{D}_B \Psi_B + \mathbf{P}_a$
- Apply the $\mathbf{D}^\dagger_{E(B)}$ to $\mathbf{P}$: $\mathbf{D}^\dagger_E \mathbf{P} = \mathbf{D}^\dagger_E \mathbf{D}_E \Psi_E$ and $\mathbf{D}^\dagger_B \mathbf{P} = \mathbf{D}^\dagger_B \mathbf{D}_B \Psi_B$
  because $\mathbf{D}^\dagger_E \mathbf{D}_B = \mathbf{D}^\dagger_B \mathbf{D}_E = 0$ and $\mathbf{D}^\dagger_E \mathbf{P}_a = \mathbf{D}^\dagger_B \mathbf{P}_a = 0$ by construction

Filtering the ambiguous modes

Searching for $\Psi_{E(B)}$ fields

- $Q + iU = \eth\eth(\Psi_E + i\Psi_B)$ and $Q - iU = \bar{\eth}\bar{\eth}(\Psi_E - i\Psi_B)$

$$2\Psi_E = \bar{\eth}\bar{\eth}(Q + iU) + \eth\eth(Q - iU) = -2\sum\sqrt{\frac{(l-2)!}{(l+2)!}}a^E_{lm}Y_{lm}$$
$$2\Psi_B = -i\bar{\eth}\bar{\eth}(Q + iU) + i\eth\eth(Q - iU) = -2\sum\sqrt{\frac{(l-2)!}{(l+2)!}}a^B_{lm}Y_{lm}$$

Working with Ψ fields

- If Ψ fields are measured, the EB mixing is completely removed
- If Stokes parameters are measured, then taking derivatives of pixelised maps leads to (controled) EB mixing and to noise with a very blue spectrum $\mathcal{N}_\ell \propto \ell^4$

Filtering the ambiguous modes

The origin of E mode into B mode

- the B multipole is the inner product of pure B mode spherical harmonics with polarization fields :
$\tilde{a}^B_{lm} = \int d\vec{n} W(\vec{n}) \mathbf{D}^\dagger_B Y^\dagger_{lm} \cdot \mathbf{P}$
- $\mathbf{P}$ is decomposed into pure E, pure B and ambiguous modes :

$$\begin{aligned}
\tilde{a}^B_{lm} &= \underbrace{\int d\vec{n} W(\vec{n}) \mathbf{D}^\dagger_B Y^\dagger_{lm} \cdot \mathbf{D}_E \Psi_E}_{=0} \\
&+ \underbrace{\int d\vec{n} W(\vec{n}) \mathbf{D}^\dagger_B Y^\dagger_{lm} \cdot \mathbf{D}_B \Psi_B}_{\neq 0} \\
&+ \underbrace{\int d\vec{n} W(\vec{n}) \mathbf{D}^\dagger_B Y^\dagger_{lm} \cdot \mathbf{P}_a}_{\neq 0}
\end{aligned}$$

Filtering the ambiguous modes

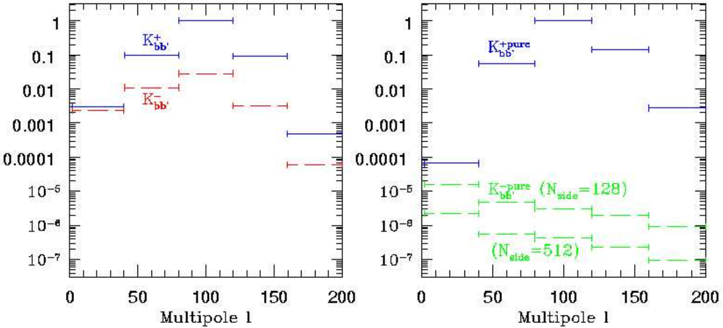
Use a pure B spherical harmonics on the incomplete sky

- New definition of the multipole estimators :
  $\tilde{a}^B_{lm} = \int d\vec{n} \mathbf{D}^\dagger_B (W \times Y^\dagger_{lm}) \cdot \mathbf{P}$
  with W satisfying **Dirichlet and Neuman** boundary conditions
- $\mathbf{P}$ is decomposed into pure E, pure B and ambiguous modes :

$$\begin{aligned}\tilde{a}^B_{lm} &= \underbrace{\int d\vec{n} \mathbf{D}^\dagger_B (W Y^\dagger_{lm}) \cdot \mathbf{D}_E \Psi_E}_{=0} \\ &+ \underbrace{\int d\vec{n} \mathbf{D}^\dagger_B (W Y^\dagger_{lm}) \cdot \mathbf{D}_B \Psi_B}_{\neq 0} \\ &+ \underbrace{\int d\vec{n} \mathbf{D}^\dagger_B (W Y^\dagger_{lm}) \cdot \mathbf{P}_a}_{=0}\end{aligned}$$

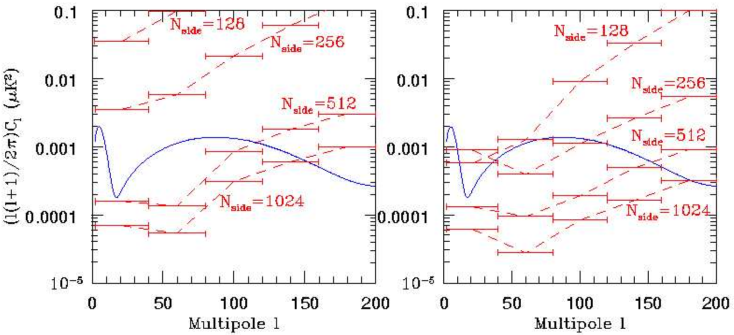
Filtering the ambiguous modes: some results

Spherical cap of 13° with uniform white noise



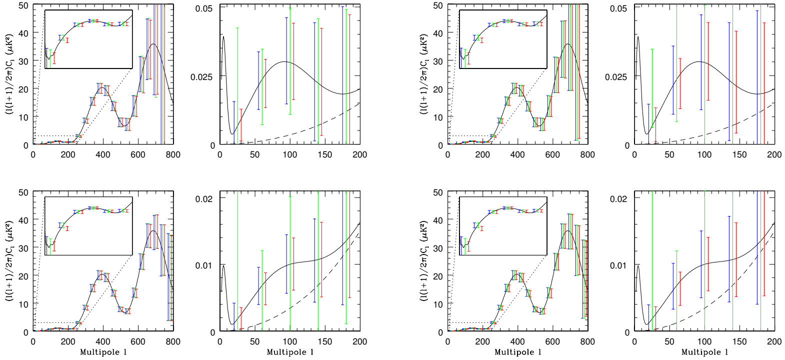
Filtering the ambiguous modes: some results

Spherical cap of 13° with uniform white noise



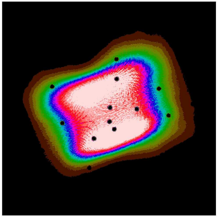
Filtering the ambiguous modes: some results

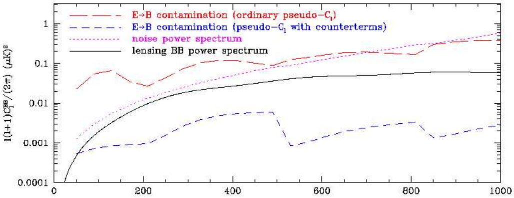
Spherical cap of 13° with uniform noise (left) or inhomogeneous noise (right)



Filtering the ambiguous modes: with optimized window function

EBEx type experiment with homogeneous uncorrelated noise: aliased power





Filtering the ambiguous modes: with optimized window function

EBEx type experiment with homogeneous uncorrelated noise: error bars