INSTITUT D'ASTROPHYSIQUE DE PARIS
Unité mixte de recherche 7095 CNRS - Université Pierre et Marie Curie

Journés Scientifiques de l'Action Spécifique GRAM

Gravitation, Références, Astronomie, Métrologie

GRAVITATIONAL WAVES and PROBLEM OF MOTION IN GR

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Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris

2 Juin 2016

100 years of gravitational radiation [Einstein 1916]

348 DOC. 32 INTEGRATION OF FIELD EQUATIONS

688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable $x_4 = it$ aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \qquad (1)$$

⟸ small perturbation of Minkowski's metric

definierten Größen $\gamma_{\mu\nu}$, welche linearen orthogonalen Transformationen gegenüber Tensorcharakter besitzen, gegen 1 als kleine Größen behandelt werden können, deren Quadrate und Produkte gegen die ersten Potenzen vernachlässigt werden dürfen. Dabei ist $\delta_{\mu\nu} = 1$ bzw. $\delta_{\mu\nu} = 0$, je nachdem $\mu = \nu$ oder $\mu \neq \nu$.

Wir werden zeigen, daß diese $\gamma_{\mu\nu}$ in analoger Weise berechnet werden können wie die retardierten Potentiale der Elektrodynamik.

100 years of gravitational radiation [Einstein 1918]

Einstein's quadrupole formula

mit $4\pi R^2$ multiplizierte $\bar{S}$ endlich ist der Energieverlust pro Zeiteinheit des mechanischen Systems durch Gravitationswellen. Die Rechnung ergibt

[31] $$4\pi R^2 \bar{S} = \frac{\varkappa}{80\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^2\right]. \qquad (30)$$

Man sieht an diesem Ergebnis, daß ein mechanisches System, welches dauernd Kugelsymmetrie behält, nicht strahlen kann, im Gegensatz
[32] zu dem durch einen Rechenfehler entstellten Ergebnis der früheren Abhandlung.

Aus (27) ist ersichtlich, daß die Ausstrahlung in keiner Richtung negativ werden kann, also sicher auch nicht die totale Ausstrahlung. Bereits in der früheren Abhandlung ist betont geworden, daß das Endergebnis dieser Betrachtung, welches einen Energieverlust der Körper infolge der thermischen Agitation verlangen würde, Zweifel an der allgemeinen Gültigkeit der Theorie hervorrufen muß. Es scheint, daß eine vervollkommnete Quantentheorie eine Modifikation auch der Gra-
[33] vitationstheorie wird bringen müssen.

§ 5 Einwirkung von Gravitationswellen auf mechanische Systeme.

Der Vollständigkeit halber wollen wir auch kurz überlegen, inwiefern Energie von Gravitationswellen auf mechanische Systeme übergehen kann. Es liege wieder ein mechanisches System vor von der

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factor 1/80 should be 1/40

§ 5 Einwirkung von Gravitationswellen auf mechanische Systeme.

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Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]

1. First quadrupole formula

$$h_{ij}^{\mathsf{TT}} = \frac{2G}{c^4 D}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t - \frac{D}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\mathsf{TT}} + \mathcal{O}\left(\frac{1}{D^2}\right)$$

2. Einstein quadrupole formula

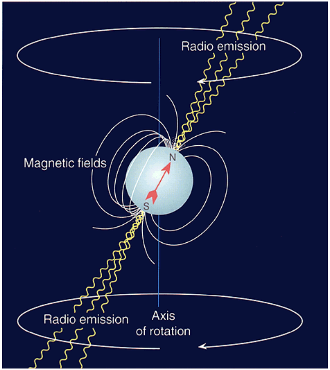
$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\mathsf{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}$$

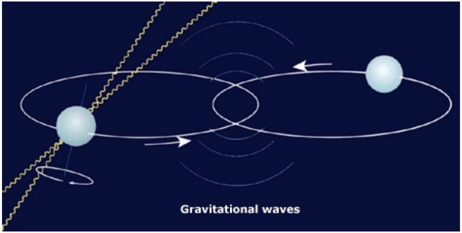
3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$F_i^{\mathsf{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

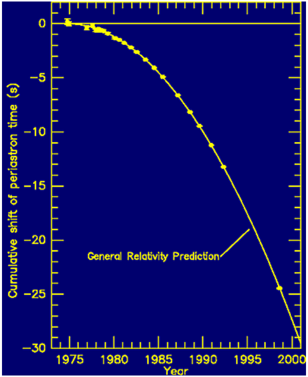
The binary pulsar PSR 1913+16 [Hulse & Taylor 1974]





- The pulsar PSR 1913+16 is a rapidly rotating neutron star emitting radio waves like a lighthouse toward the Earth
- This pulsar moves on a (quasi-)Keplerian close orbit around an unseen companion, probably another neutron star

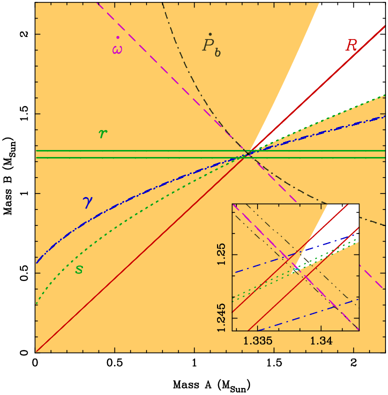
The orbital decay of the binary pulsar [Taylor & Weisberg 1982]



$$\dot{P} = -\frac{192\pi}{5c^5}\nu\left(\frac{2\pi G\,M}{P}\right)^{5/3}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} \approx -2.4\times 10^{-12}$$

[Peters & Mathews 1963; Esposito & Harrison 1975; Wagoner 1975; Damour & Deruelle 1983]

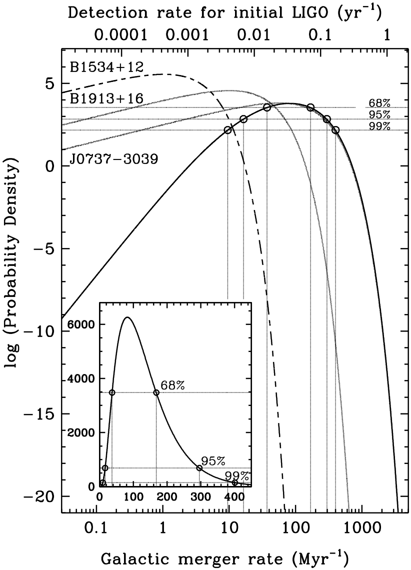
Relativistic effects in binary pulsars [e.g. Stairs 2003]



1PN order $\left\{ \begin{array}{l} \dot{\omega} \text{ relativistic advance of periastron} \\ \gamma \text{ gravitational red-shift and second-order Doppler effect} \\ r \text{ and } s \text{ range and shape of the Shapiro time delay} \end{array} \right.$

2.5PN order $\{$ $\dot{P}$ secular decrease of orbital period

Number of merging neutron star binaries [Kalogera *et al.* 2004]



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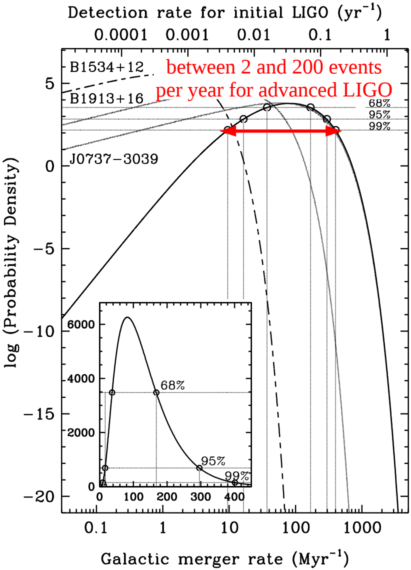
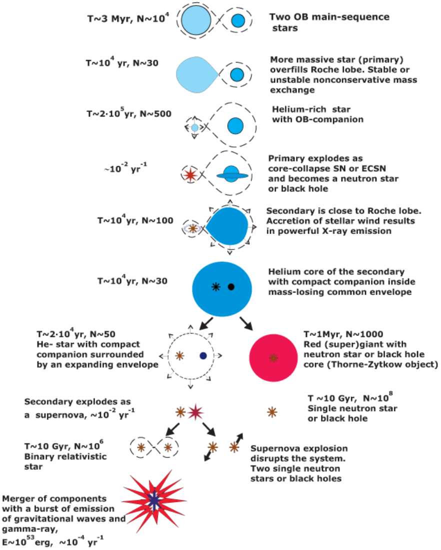
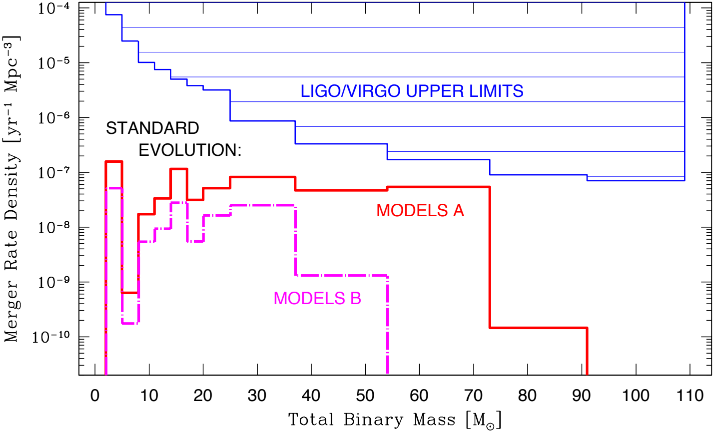


Formation of black hole binaries [Postnov & Yungelson 2006]



Number of merging black hole binaries [Belczynski *et al.* 2014]



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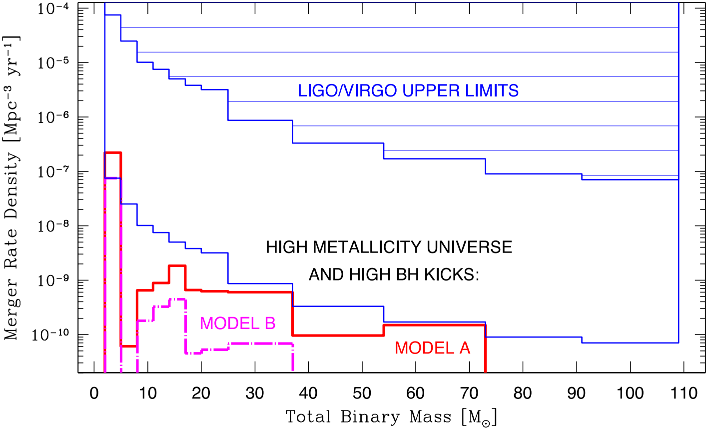
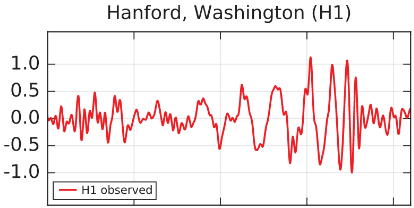
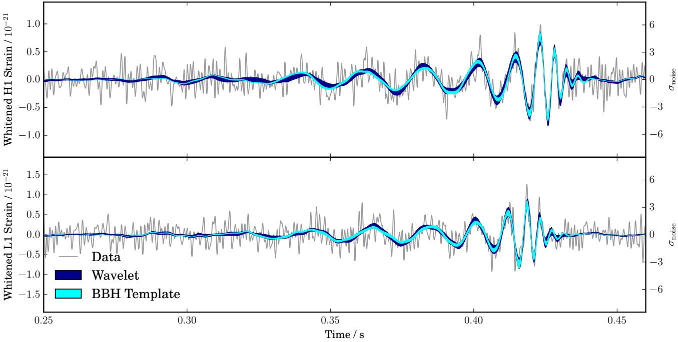


Binary black-hole event GW150914 [LIGO/VIRGO collaboration 2016]





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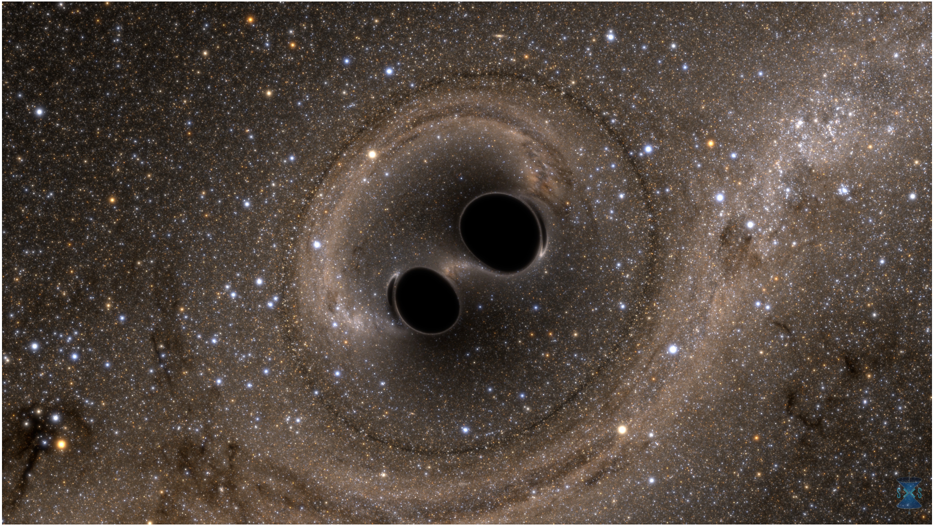

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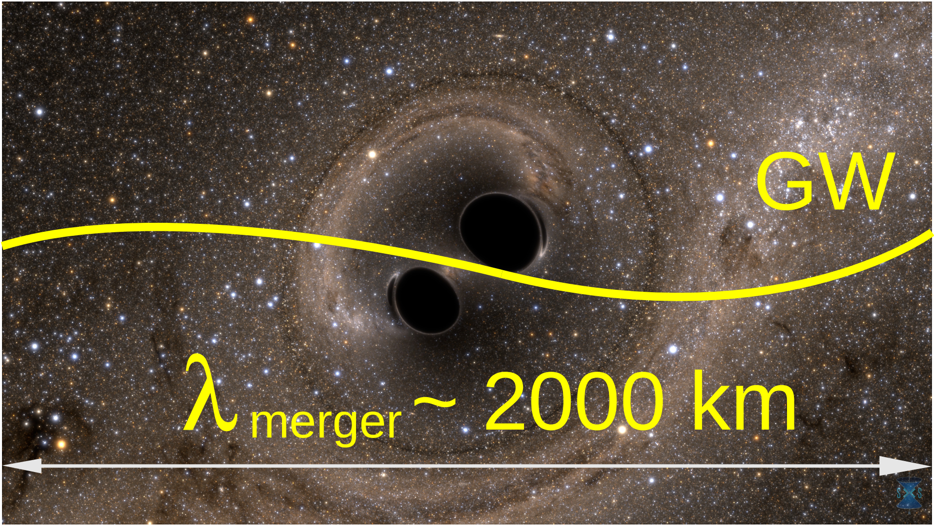


The quadrupole formula works for GW150914

1. The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi}\left[\frac{256}{5}\frac{G\mathcal{M}^{5/3}}{c^5}(t_\mathsf{f} - t)\right]^{-3/8}$$

2. Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96}\frac{c^5}{G\pi^{8/3}} f^{-11/3}\dot{f}\right]^{3/5}}$$

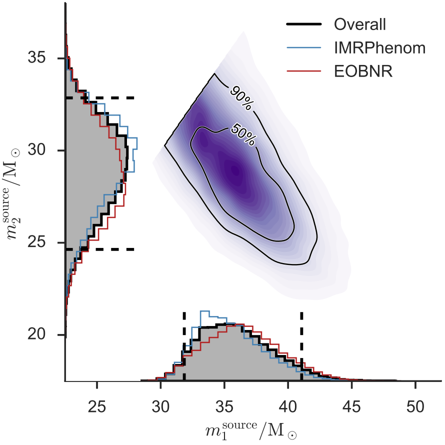
which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

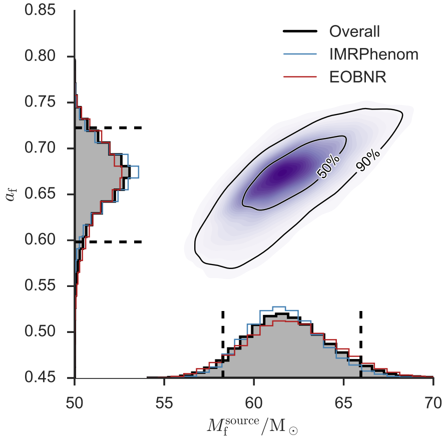
3. The GW amplitude is predicted to be

$$h_\text{eff} \sim 4.1 \times 10^{-22}\left(\frac{\mathcal{M}}{M_\odot}\right)^{5/6}\left(\frac{100\,\text{Mpc}}{D}\right)\left(\frac{100\,\text{Hz}}{f_\text{merger}}\right)^{-1/6} \sim 1.6 \times 10^{-21}$$

4. The distance $D = 400\,\text{Mpc}$ is measured from the signal itself

Masses & spin measurements [LIGO/VIRGO collaboration 2016]





Total energy radiated by GW150914

1. The ADM energy of space-time is constant and reads (at any t)

$$E_{\mathsf{ADM}} = (m_1+m_2)c^2 - \frac{G m_1 m_2}{2r} + \frac{G}{5c^5}\int_{-\infty}^{t} \mathrm{d}t' \big(Q^{(3)}_{ij}\big)^2(t')$$

2. Initially $E_{\mathsf{ADM}} = (m_1+m_2)c^2$ while finally (at time t_{f})

$$E_{\mathsf{ADM}} = M_{\mathsf{f}} c^2 + \frac{G}{5c^5}\int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \big(Q^{(3)}_{ij}\big)^2(t')$$

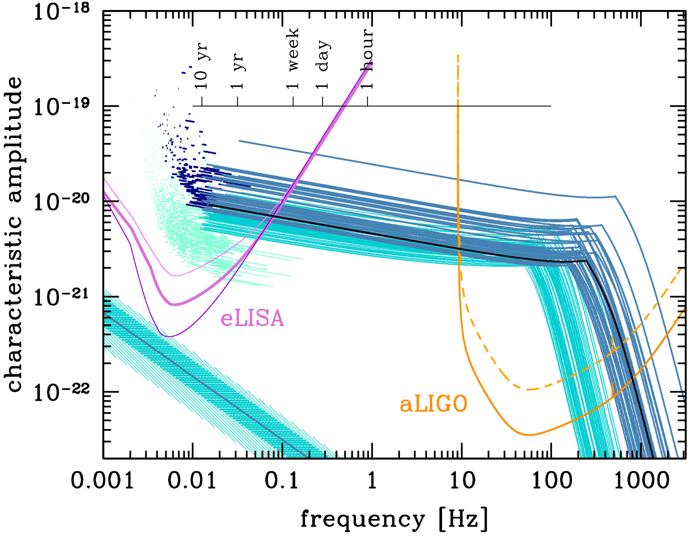
3. The total energy radiated in GW is

$$\boxed{\Delta E^{\mathsf{GW}} = \frac{G}{5c^5}\int_{-\infty}^{t_{\mathsf{f}}} \mathrm{d}t' \big(Q^{(3)}_{ij}\big)^2(t') = \frac{G m_1 m_2}{2 r_{\mathsf{f}}}}$$

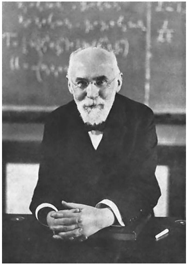
4. The measured power released is

$$\boxed{P^{\mathsf{GW}} \sim \frac{3 M_\odot c^2}{0.2\,\mathsf{s}} \sim 10^{55}\,\mathsf{erg/s} \sim 10^{-4}\,\frac{c^5}{G}}$$

Multi-band gravitational wave astronomy [Sesana 2016]

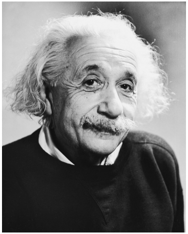


The 1PN equations of motion [Lorentz & Droste 1917]



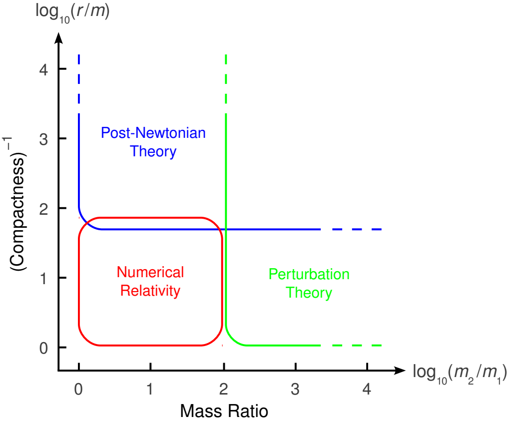
- Obtain the equations of motion of N bodies at the 1PN $\sim (v/c)^2$ order and even derive the 1PN Lagrangian!
- This work published in Dutch has been largely unrecognized untill an English translation was published in 1937

The 1PN equations of motion [Einstein, Infeld & Hoffmann 1938]



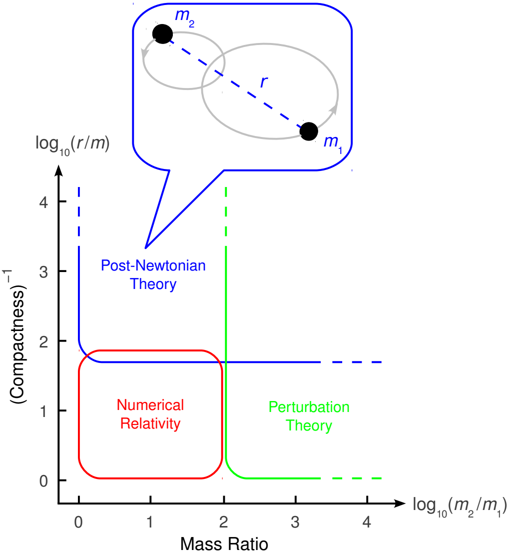
$$\frac{\mathrm{d}^2\boldsymbol{r}_A}{\mathrm{d}t^2} = -\sum_{B\neq A}\frac{Gm_B}{r_{AB}^2}\boldsymbol{n}_{AB}\Bigg[1 - 4\sum_{C\neq A}\frac{Gm_C}{c^2 r_{AC}} - \sum_{D\neq B}\frac{Gm_D}{c^2 r_{BD}}\left(1 - \frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BD}}{r_{BD}^2}\right) + \frac{1}{c^2}\left(\boldsymbol{v}_A^2 + 2\boldsymbol{v}_B^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_B - \frac{3}{2}(\boldsymbol{v}_B\cdot\boldsymbol{n}_{AB})^2\right)\Bigg] + \sum_{B\neq A}\frac{Gm_B}{c^2 r_{AB}^2}\boldsymbol{v}_{AB}[\boldsymbol{n}_{AB}\cdot(3\boldsymbol{v}_B - 4\boldsymbol{v}_A)] - \frac{7}{2}\sum_{B\neq A}\sum_{D\neq B}\frac{G^2 m_B m_D}{c^2 r_{AB} r_{BD}^3}\boldsymbol{n}_{BD}$$

Methods to compute GW templates



[courtesy Alexandre Le Tiec]

Methods to compute GW templates



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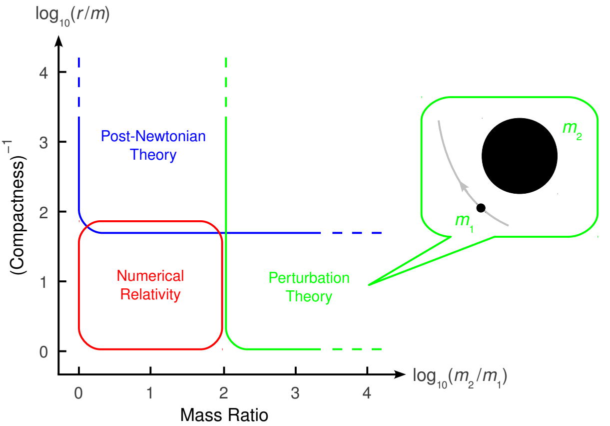
Methods to compute GW templates



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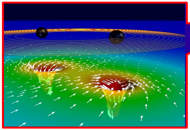
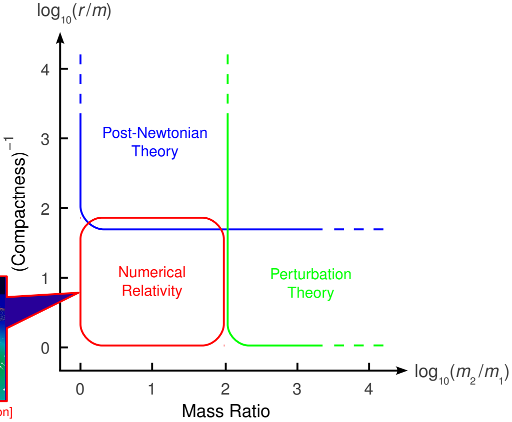
Methods to compute GW templates

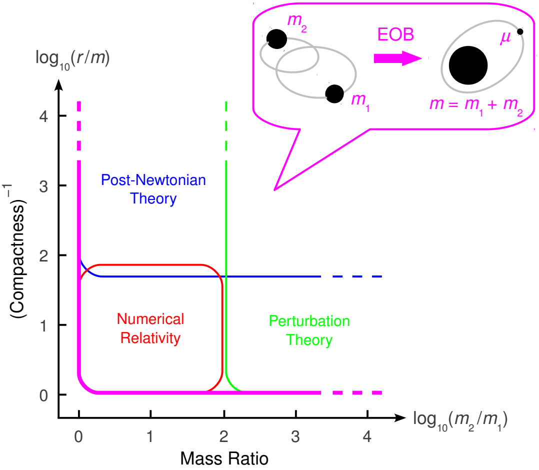




[courtesy Alexandre Le Tiec]

Methods to compute GW templates

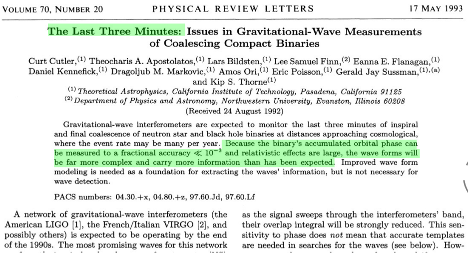
[Buonanno & Damour 1998]



[courtesy Alexandre Le Tiec]

Inspiralling binaries require high-order PN modelling

[Caltech "3mn paper" 1992; Blanchet & Schäfer 1993]



VOLUME 70, NUMBER 20 PHYSICAL REVIEW LETTERS 17 MAY 1993

The Last Three Minutes: Issues in Gravitational-Wave Measurements of Coalescing Compact Binaries

Curt Cutler,[1] Theocharis A. Apostolatos,[1] Lars Bildsten,[1] Lee Samuel Finn,[2] Eanna E. Flanagan,[1] Daniel Kennefick,[1] Dragoljub M. Markovic,[1] Amos Ori,[1] Eric Poisson,[1] Gerald Jay Sussman,[1],[a] and Kip S. Thorne[1]

[1] *Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125*
[2] *Department of Physics and Astronomy, Northwestern University, Evanston, Illinois 60208*

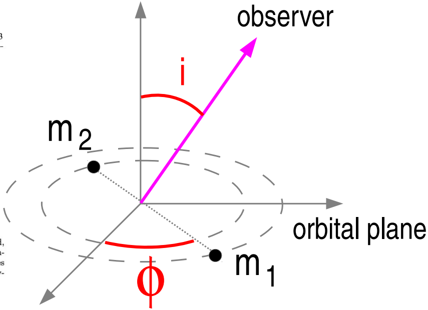
(Received 24 August 1992)

Gravitational-wave interferometers are expected to monitor the last three minutes of inspiral and final coalescence of neutron star and black hole binaries at distances approaching cosmological, where the event rate may be many per year. Because the binary's accumulated orbital phase can be measured to a fractional accuracy $\ll 10^{-3}$ and relativistic effects are large, the wave forms will be far more complex and carry more information than has been expected. Improved wave form modeling is needed as a foundation for extracting the waves' information, but is not necessary for wave detection.

PACS numbers: 04.30.+x, 04.80.+z, 97.60.Jd, 97.60.Lf

A network of gravitational-wave interferometers (the American LIGO [1], the French/Italian VIRGO [2], and possibly others) is expected to be operating by the end of the 1990s. The most promising waves for this network

as the signal sweeps through the interferometers' band, their overlap integral will be strongly reduced. This sensitivity to phase does *not* mean that accurate templates are needed in searches for the waves (see below). How-



$$\phi(t) = \phi_0 - \underbrace{\frac{M}{\mu}\left(\frac{GM\omega}{c^3}\right)^{-5/3}}_{\text{quadrupole formalism}} \Big\{ 1 + \underbrace{\frac{1\text{PN}}{c^2} + \frac{1.5\text{PN}}{c^3} + \cdots + \frac{3\text{PN}}{c^6} + \cdots}_{\text{needs to be computed with 3PN precision at least}} \Big\}$$

Here 3PN means 5.5PN as a radiation reaction effect !

The intermediate binary black hole problem

PHYSICAL REVIEW D, VOLUME 58, 061501

Computing the merger of black-hole binaries: The IBBH problem

Patrick R. Brady, Jolien D. E. Creighton, and Kip S. Thorne
Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125
(Received 22 April 1998; published 26 August 1998)

Gravitational radiation arising from the inspiral and merger of binary black holes (BBH's) is a promising candidate for detection by kilometer-scale interferometric gravitational wave observatories. This Rapid Communication discusses a serious obstacle to searches for such radiation and to the interpretation of any observed waves: the inability of current computational techniques to evolve a BBH through its last ~10 orbits of inspiral (~100 radians of gravitational-wave phase). A new set of numerical-relativity techniques is proposed for solving this ''intermediate binary black hole'' (IBBH) problem: (i) numerical evolutions performed in coordinates co-rotating with the BBH, in which the metric coefficients evolve on the long timescale of inspiral, and (ii) techniques for mathematically freezing out gravitational degrees of freedom that are not excited by the waves. [S0556-2821(98)50218-4]

PACS number(s): 04.25.Dm, 04.30.Db, 04.70.−s

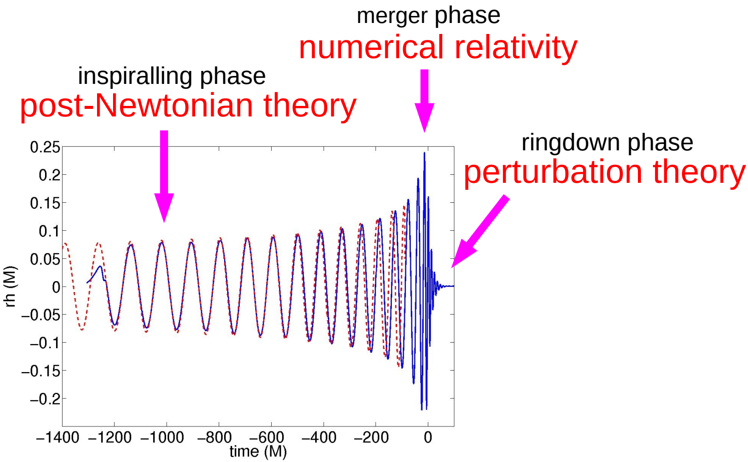
I. MOTIVATION

Among all gravitational wave sources that theorists have considered, the one most likely to be detected first is the final inspiral and merger of binary black holes (BBH's) with

that, in the next several years, this approach will be able to evolve a BBH through the gap for the required $\gtrsim$1200 dynamical time scales. This motivates exploring alternative procedures for computing the evolution and waves during the IBBH phase.

- An alternative solution is to extend the region of validity of the PN approximation by using Padé approximants [Damour, Iyer & Sathyaprakash 1998]
- However the accuracy of the PN approximation for comparable masses turned out to be very good rather far into the strong field region [Blanchet 2001]

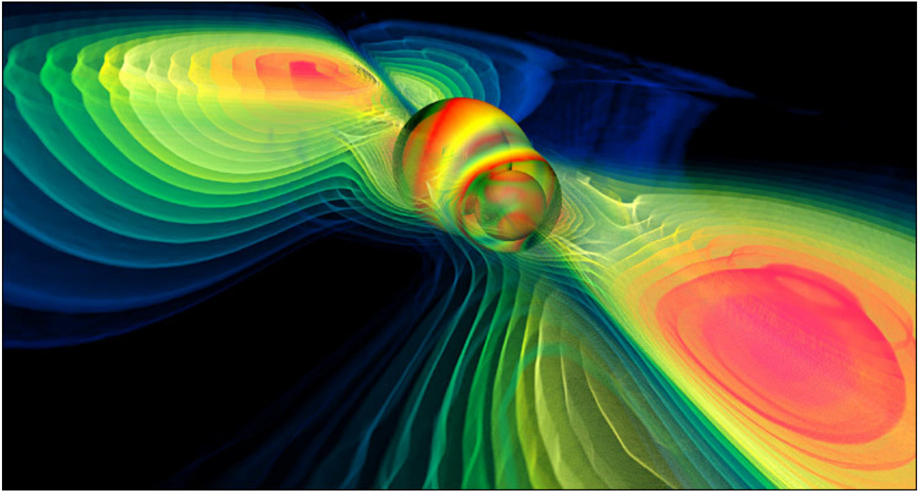
The gravitational chirp of compact binaries



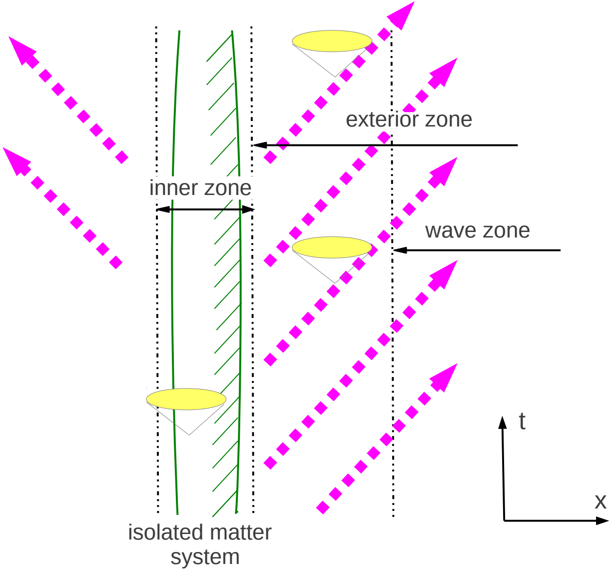
Effective methods such as EOB that interpolate between the PN and NR are very important notably for the data analysis

Breakthrough of numerical relativity

[Pretorius 2005; Baker *et al.* 2006; Campanelli *et al.* 2006]



Isolated matter system in general relativity



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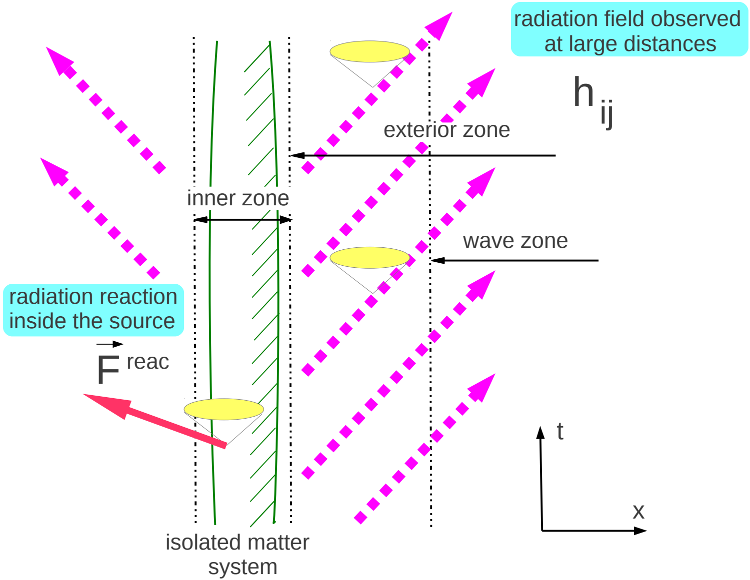
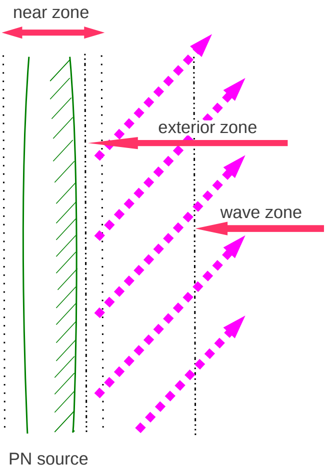


The MPM-PN formalism

[Blanchet & Damour 1986, 1988, 1992; Blanchet 1987, 1993, 1998]

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



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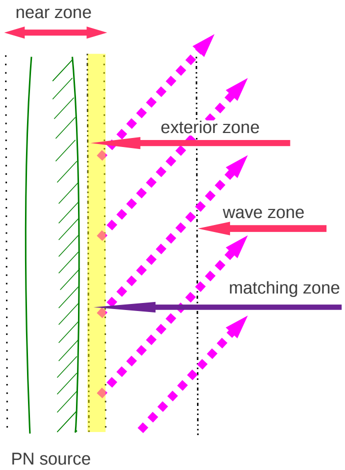


Radiative moments at future null infinity

Correct for the logarithmic deviation of retarded time in harmonic coordinates with respect to the actual null coordinate

$$\underbrace{T - \frac{R}{c}}_{\text{radiative coordinates}} = \underbrace{t - \frac{r}{c}}_{\text{harmonic coordinates}} - \frac{2GM}{c^3} \ln\left(\frac{r}{c\tau_0}\right) + \mathcal{O}\left(\frac{1}{r}\right)$$

Asymptotic waveform is parametrized by radiative moments U_L and V_L [Thorne 1980]

$$h_{ij}^{\mathsf{TT}} = \frac{1}{R} \sum_{\ell=2}^{\infty} N_{L-2} \underbrace{U_{ijL-2}(T - R/c)}_{\text{mass-type}} + \epsilon_{ab(i} N_{aL-1} \underbrace{V_{j)bL-2}(T - R/c)}_{\text{current-type}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

The radiative quadrupole moment

$$U_{ij}(t) = M^{(2)}_{ij}(t) + \underbrace{\frac{2GM}{c^3}\int_0^{+\infty} d\tau\, M^{(4)}_{ij}(t-\tau)\left[\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{12}\right]}_{\text{1.5PN tail integral}}$$
$$+ \frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty} d\tau\, M^{(3)}_{a<i}M^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\}$$
$$+ \underbrace{\frac{2G^2M^2}{c^6}\int_0^{+\infty} d\tau\, M^{(5)}_{ij}(t-\tau)\left[\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{70}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{44100}\right]}_{\text{3PN tail-of-tail integral [Blanchet 1998]}}$$
$$+ \mathcal{O}\left(\frac{1}{c^7}\right)$$

At 4.5PN order presence of a tail-of-tail-of-tail [Marchand, Blanchet & Faye, in progress]

Dimensional self-field regularization

1. Einstein's field equations are solved in d spatial dimensions (with $d \in \mathbb{C}$) with distributional sources. In Newtonian approximation

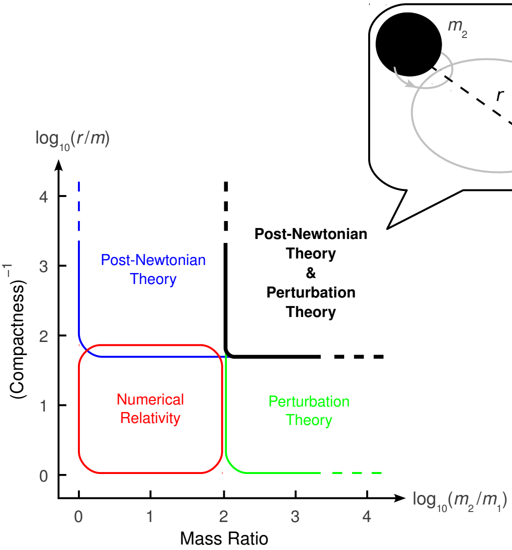
$$\Delta U = -4\pi \frac{2(d-2)}{d-1} G\rho$$

2. For two point-particles $\rho = m_1 \delta_{(d)}(\mathbf{x} - \mathbf{y}_1) + m_2 \delta_{(d)}(\mathbf{x} - \mathbf{y}_2)$ we get

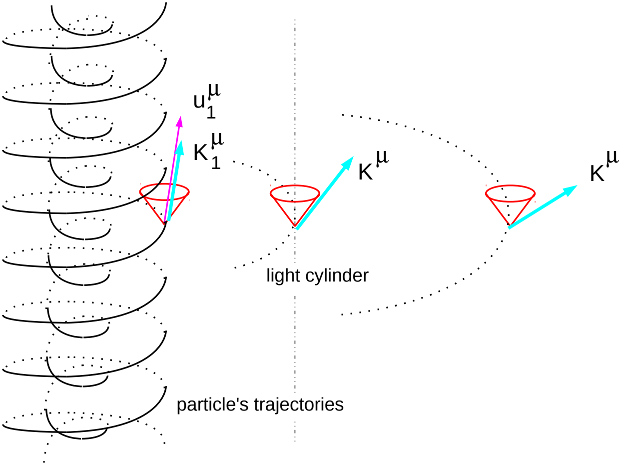
$$\boxed{U(\mathbf{x},t) = \frac{2(d-2)k}{d-1}\left(\frac{G m_1}{|\mathbf{x}-\mathbf{y}_1|^{d-2}} + \frac{G m_2}{|\mathbf{x}-\mathbf{y}_2|^{d-2}}\right) \quad \text{with} \quad k = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}}$$

3. Computations are performed when $\Re(d)$ is a large negative number, and the result is analytically continued for any $d \in \mathbb{C}$ except for isolated poles
4. Dimensional regularization is then followed by a renormalization of the worldline of the particles so as to absorb the poles $\propto (d-3)^{-1}$

Checking the PN machinery with GSF



Looking at the conservative part of the dynamics



$$u_1^\mu = u_1^t K^\mu \quad \text{where} \quad u_1^t = \frac{1}{z_1} = \Bigg(- \underbrace{(g_{\mu\nu})_1}_{\text{regularized metric}} \frac{v_1^\mu v_1^\nu}{c^2} \Bigg)^{-1/2}$$

[Detweiler 2008]

Standard PN theory agrees with GSF calculations

$$
\begin{aligned}
u^t_{\mathrm{SF}} = &-y - 2y^2 - 5y^3 + \left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4 \\
&+ \left(-\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma_\mathsf{E} - \frac{64}{5}\ln(16y)\right) y^5 \\
&-\frac{956}{105} y^6 \ln y - \frac{13696\pi}{525} y^{13/2} - \frac{51256}{567} y^7 \ln y + \frac{81077\pi}{3675} y^{15/2} \\
&+ \frac{27392}{525} y^8 \ln^2 y + \frac{82561159\pi}{467775} y^{17/2} - \frac{27016}{2205} y^9 \ln^2 y \\
&- \frac{11723776\pi}{55125} y^{19/2} \ln y - \frac{4027582708}{9823275} y^{10} \ln^2 y \\
&+ \frac{99186502\pi}{1157625} y^{21/2} \ln y + \frac{23447552}{165375} y^{11} \ln^3 y + \cdots
\end{aligned}
$$

- Integral PN terms such as 3PN permit checking dimensional regularization [Blanchet, Detweiler, Le Tiec & Whiting 2010]
- Half-integral PN terms starting at 5.5PN order permit checking the non-linear tail (and tail-of-tail) terms [Blanchet, Faye & Whiting 2014]

4PN equations of motion of compact binaries

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2} n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \cdots\right] n_{12}^i + \cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

2PN:

[Otha, Okamura, Kimura & Hiida 1973, 1974; Damour & Schäfer 1985]	ADM Hamiltonian
[Damour & Deruelle 1981; Damour 1983]	Harmonic coordinates
[Kopeikin 1985; Grishchuk & Kopeikin 1986]	Extended fluid balls
[Blanchet, Faye & Ponsot 1998]	Direct PN iteration
[Itoh, Futamase & Asada 2001]	Surface integral method

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3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001]	ADM Hamiltonian
	[Blanchet & Faye 2000; de Andrade, Blanchet & Faye 2001]	Harmonic equations of motion
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye & Marsat 2015]	Fokker Lagrangian

3.5PN energy flux of compact binaries

[BDIWW 1995; B 1996, 1998; BFIJ 2002; BDEI 2006]

$$\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 + \overbrace{\left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x}^{\text{1PN}} + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}}$$
$$+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \overbrace{\left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2}}^{\text{2.5PN tail}}$$
$$+ \Bigg[\frac{6643739519}{69854400} + \overbrace{\frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\ln(16\,x)}^{\text{3PN tail-of-tail}}$$
$$+ \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3 \Bigg] x^3$$
$$+ \underbrace{\left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2}}_{\text{3.5PN tail}} + \mathcal{O}\left(\frac{1}{c^8}\right) \Bigg\}$$

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Bounds on PN parameters with GW150914

[LIGO/VIRGO collaboration 2016]