Cosmic Ray Small-Scale Anisotropies

Philipp Mertsch
with Markus Ahlers

"Searching for the Sources of Galactic Cosmic Rays"
Paris, 13 December 2018

Outline

1. Observations
2. Quasi-linear theory
3. Small-scale turbulence model
4. Other models for small-scale anisotropies
   - Magnetic lenses etc.
   - Non-uniform pitch-angle scattering
   - Heliospheric effects

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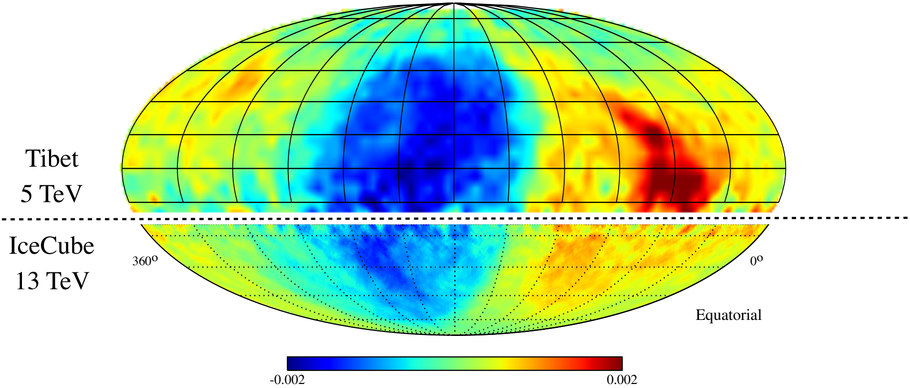
Cosmic ray anisotropies

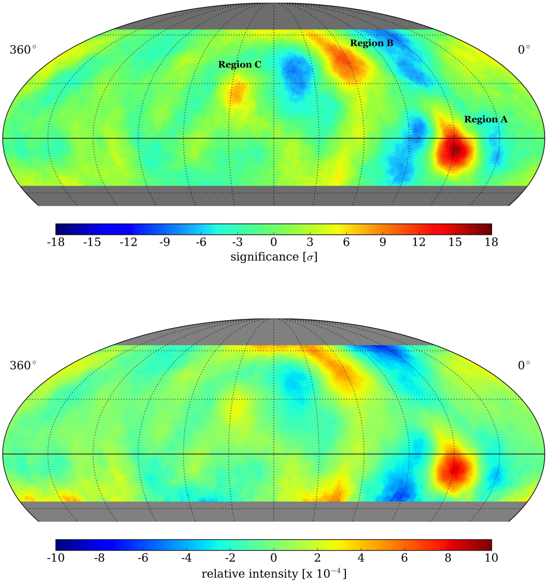
$$I(\mathbf{n}) \equiv \frac{\phi(\mathbf{n})}{\phi^{\text{iso}}} \equiv 1 + \delta I(\mathbf{n})$$



Amenomori *et al.*, ApJ 711 (2010) 119, Saito *et al.*, *Proc. 32nd ICRC* 1 (2011) 62

Aartsen *et al.*, ApJ 826 (2016) 220

Small-scale anisotropies

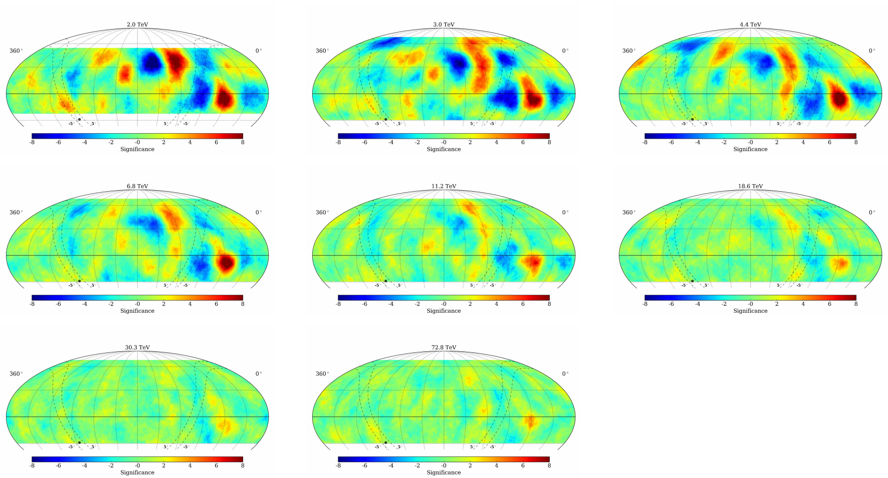


- subtract off dipole and quadrupole
- smooth with 10° disk

→ small-scale features

Abeysekara *et al.*, ApJ 796 (2014) 108

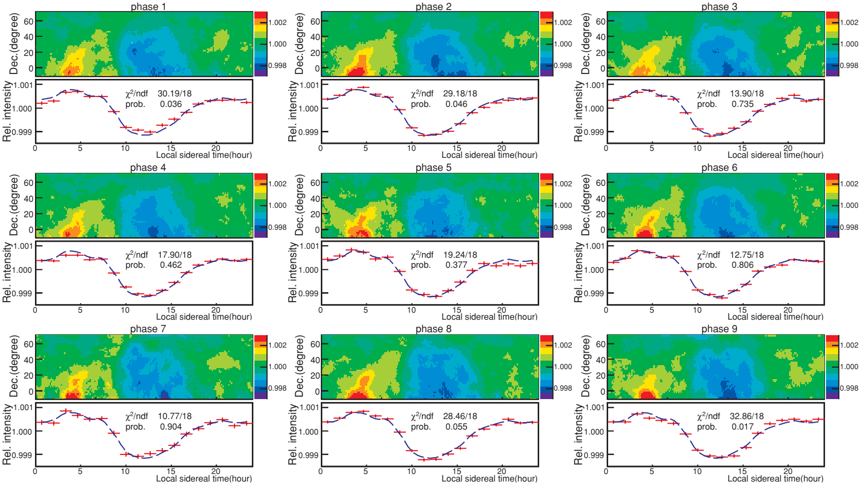
Energy dependence



Abeysekara *et al.*, arXiv:1805.01847

Decrease of amplitude and flip of direction around 100 TeV also seen by IceCube

Time dependence



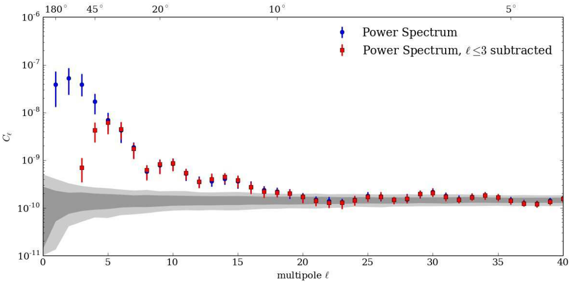
Amenomori *et al.*, ApJ 711 (2010) 119

No significant time-dependence over 9 years.

Angular power spectrum

HAWC

Abeysekara *et al.*, ApJ 796 (2014) 108
also Abeysekara *et al.*, arXiv:1805.01847

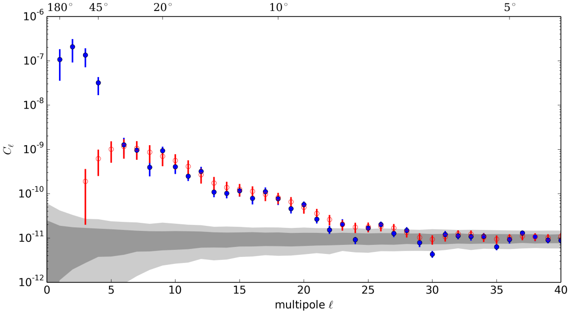


IceCube

Aartsen *et al.*, ApJ 826 (2016) 220

IceCube+HAWC

Daz-Vlez *et al.*, arXiv:1708.03005



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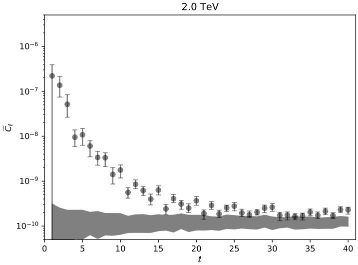

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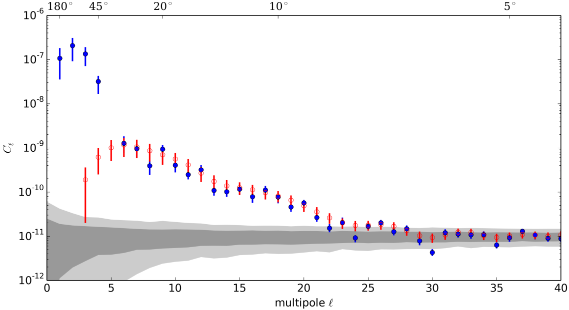

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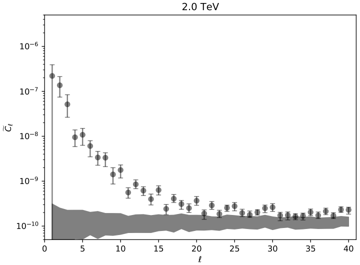

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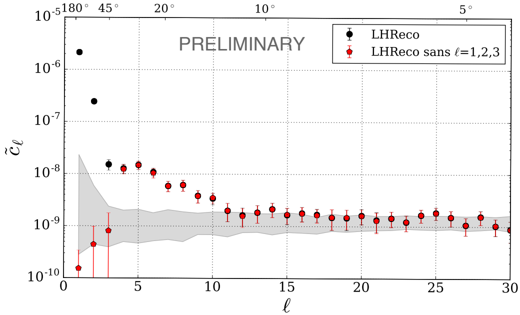


Score sheet

Properties

- large-scale anisotropy of the order $10^{-3} \dots 10^{-4}$ at TeV . . . PeV energies
- small-scale anisotropy of similar size
- directional pattern also changes with energy
- no time-dependence

Limitation

Relative intensity in declination bands not fixed by reconstruction
$\rightarrow$ insensitive to anisotropies that align with Earth's rotation axis

Outline

Vlasov equation

- Liouville's theorem:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \nabla_{\mathbf{r}} f + \dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}} f = 0$$

- In a regular and turbulent magnetic field:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \delta\mathbf{B}(\mathbf{r}) \equiv p_0/e \left(\boldsymbol{\Omega} + \boldsymbol{\omega}(\mathbf{r})\right)$$

- Angular momentum operator $\mathbf{L} \equiv -\imath \mathbf{p} \times \nabla_{\mathbf{p}}$:

$$\dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}} f = \mathbf{p} \times (\boldsymbol{\Omega} + \boldsymbol{\omega}(\mathbf{r})) \cdot \nabla_{\mathbf{p}} f = -\imath(\boldsymbol{\Omega} + \boldsymbol{\omega}(\mathbf{r})) \cdot \mathbf{L} f$$

- Deterministic and stochastic operators $\mathcal{L}_0$ and $\delta\mathcal{L}$:

$$\frac{\partial f}{\partial t} + \underbrace{\left(\dot{\mathbf{r}} \cdot \nabla_{\mathbf{r}} - \imath \boldsymbol{\Omega} \cdot \mathbf{L}\right)}_{\mathcal{L}_0} f + \underbrace{\left(-\imath \boldsymbol{\omega} \cdot \mathbf{L}\right)}_{\delta\mathcal{L}} f = 0$$

Quasi-linear theory

e.g. Jokipii, *Rev. Geophys.* **9** (1971) 27

- Equations for averaged phase space density and fluctuations: $f = \langle f \rangle + \delta f$

$$\frac{\partial}{\partial t}\langle f \rangle + \mathcal{L}_0 \langle f \rangle = - \langle \delta \mathcal{L} \delta f \rangle \, ,$$
$$\frac{\partial}{\partial t}\delta f + \mathcal{L}_0 \delta f \simeq - \delta \mathcal{L} \langle f \rangle \, .$$

- Integration along *unperturbed trajectories* $P(t')$

$$\delta f(t, \mathbf{r}, \mathbf{p}) \simeq \delta f(t_0, \mathbf{r}(t_0), \mathbf{p}(t_0)) - \int_{t_0}^{t} \mathrm{d}t' \left[\delta \mathcal{L} \langle f \rangle \right]_{P(t')}$$

- Scattering term $\langle \delta \mathcal{L} \delta f \rangle$ can be approximated as

$$\langle \delta \mathcal{L} \delta f \rangle \simeq - \left\langle \delta \mathcal{L} \int_{-\infty}^{t} \mathrm{d}t' \left[\delta \mathcal{L} \langle f \rangle \right]_{P(t')} \right\rangle \simeq \frac{\partial}{\partial \mu} D_{\mu\mu} \frac{\partial}{\partial \mu} \langle f \rangle$$

$\rightarrow$ Pitch-angle diffusion $\rightarrow$ spatial diffusion

Outline

Small-scale turbulence and ensemble averaging

- in standard diffusion, compute C_ℓ from $\langle f \rangle$:

$$C_\ell^{\text{std}} = \frac{1}{4\pi} \int \mathrm{d}\hat{\mathbf{p}}_1 \int \mathrm{d}\hat{\mathbf{p}}_2 \, P_\ell(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \langle f(\hat{\mathbf{p}}_1) \rangle \langle f(\hat{\mathbf{p}}_2) \rangle$$

- however, in an individual realisation of δB, $\delta f = f - \langle f \rangle \neq 0$

$$\langle C_\ell \rangle = \frac{1}{4\pi} \int \mathrm{d}\hat{\mathbf{p}}_1 \int \mathrm{d}\hat{\mathbf{p}}_2 \, P_\ell(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2) \rangle$$

- if $f(\hat{\mathbf{p}}_1)$ and $f(\hat{\mathbf{p}}_2)$ are correlated,

$$\langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2) \rangle \geq \langle f(\hat{\mathbf{p}}_1) \rangle \langle f(\hat{\mathbf{p}}_2) \rangle \quad \Rightarrow \quad \langle C_\ell \rangle \geq C_\ell^{\text{std}}$$

Source of the small scale anisotropies?

Giacinti & Sigl, PRL **109** (2012) 071101

Ahlers, PRL **112** (2014) 021101, Ahlers & Mertsch, ApJL 815 (2015) L2, Pohl & Rettig, *Proc. 36th ICRC* (2016) 451, López-Barquero *et al.*, ApJ 830 (2016) 19, López-Barquero *et al.* ApJ 842 (2017) 54

Gradient ansatz

- Vlasov equation:

$$\frac{\partial f}{\partial t} + \underbrace{\left(\dot{\mathbf{r}} \cdot \nabla_{\mathbf{r}} - \imath \boldsymbol{\Omega} \cdot \mathbf{L}\right)}_{\mathcal{L}_0} f + \underbrace{\left(-\imath \boldsymbol{\omega} \cdot \mathbf{L}\right)}_{\delta \mathcal{L}} f = 0$$

- Gradient ansatz:

$$f(\mathbf{r}, \hat{\mathbf{p}}) = f(\mathbf{r}_0, \hat{\mathbf{p}}) + (\mathbf{r}_0 - \mathbf{r}) \cdot \mathbf{G} \,,$$

→ Dipolar source term in the Vlasov equation:

$$\frac{\partial f}{\partial t} + \underbrace{\left(-\imath \boldsymbol{\Omega} \cdot \mathbf{L}\right)}_{\mathcal{L}_0'} f + \underbrace{\left(-\imath \boldsymbol{\omega} \cdot \mathbf{L}\right)}_{\delta \mathcal{L}} f = c \, \hat{\mathbf{p}} \cdot \mathbf{G}$$

Mixing matrices

- Formal solution of Vlasov equation:

$$f(\mathbf{r}, \mathbf{p}, t) = U_{t,t_0} f(\mathbf{r}, \mathbf{p}, t_0) + \int_{t_0}^{t} \mathrm{d}t' \, U_{t,t'} \, c \, \hat{\mathbf{p}} \cdot \mathbf{G}$$

→ Differential equation for $\langle C_\ell \rangle$,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \langle C_\ell \rangle(t) + \left(\lim_{t_0 \to t} \frac{\delta_{\ell \ell_0} - M_{\ell \ell_0}(t, t_0)}{t - t_0} \right) \langle C_{\ell_0} \rangle(t) = \frac{8\pi}{9} K |\mathbf{G}|^2 \delta_{\ell 1}}$$

where

mixing $\ell_0 \to \ell$ | sourcing ℓ

$$M_{\ell \ell_0}(t, t_0) = \frac{1}{4\pi} \int \mathrm{d}\hat{\mathbf{p}}_A \int \mathrm{d}\hat{\mathbf{p}}_B \mathrm{P}_\ell(\hat{\mathbf{p}}_A \cdot \hat{\mathbf{p}}_B) \langle U^A_{t,t_0} U^{B*}_{t,t_0} \rangle \frac{2\ell_0 + 1}{4\pi} \mathrm{P}_{\ell_0}(\hat{\mathbf{p}}_A \cdot \hat{\mathbf{p}}_B)$$

- Consider the steady-state, $\mathrm{d}\langle C_\ell \rangle / \mathrm{d}t = 0$

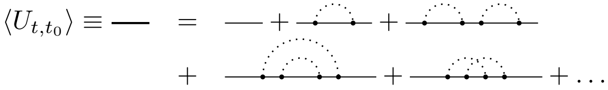
One particle propagator

"Feynman" rules

- free propagator: $U^{(0)}_{t,t'}$ 

- stochastic field: $\delta\mathcal{L}(t)$ 

- correlation: $\langle \delta\mathcal{L}(t) U^{(0)}_{t,t'} \delta\mathcal{L}(t') \rangle$ 



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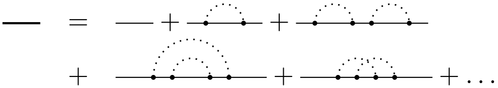
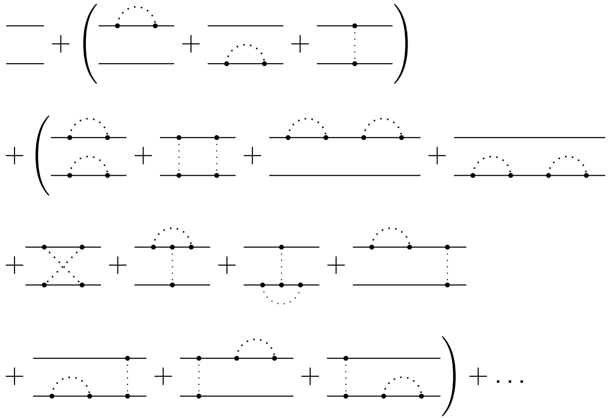
$\langle U_{t,t_0} \rangle \equiv$ 

Double propagator

For $\langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2) \rangle$ we need correlated evolution of two particles:

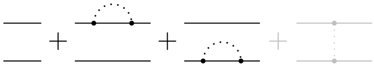
$$\langle U^{A}_{t,t_0} U^{B*}_{t,t_0} \rangle =$$



Ignoring correlations

- If $\langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2) \rangle = \langle f(\hat{\mathbf{p}}_1) \rangle \langle f(\hat{\mathbf{p}}_2) \rangle$

$$\langle U^{A}_{t,t_0} U^{B*}_{t,t_0} \rangle \quad \simeq$$



- Mixing matrix diagonal:

$$M_{\ell \ell_0}(t, t_0) \sim \delta_{\ell \ell_0}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle C_\ell \rangle (t) + \left(\lim_{t_0 \to t} \frac{\delta_{\ell \ell_0} - M_{\ell \ell_0}(t, t_0)}{t - t_0} \right) \langle C_{\ell_0} \rangle (t) = \frac{8\pi}{9} K |\mathbf{G}|^2 \delta_{\ell 1} \, ,$$

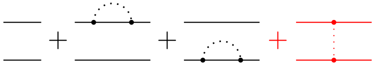
→ only dipolar anisotropy:

$$\langle C_\ell \rangle \propto \delta_{\ell 1} \, ,$$

With correlations

- $\langle f(\hat{\mathbf{p}}_1) f(\hat{\mathbf{p}}_2)\rangle \neq \langle f(\hat{\mathbf{p}}_1)\rangle\langle f(\hat{\mathbf{p}}_2)\rangle$

$$\langle U^{A}_{t,t_0} U^{B*}_{t,t_0}\rangle \simeq$$



- Mixing matrix **not** diagonal:

$$M_{\ell\ell_0}(t,t_0) \sim \delta_{\ell\ell_0} + \sum_{\ell_A} \kappa_{\ell_A}(t-t_0) \begin{pmatrix} \ell & \ell_A & \ell_0 \\ 0 & 0 & 0 \end{pmatrix}^2 (2\ell_0+1)\ell_0(\ell_0+1)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle C_\ell\rangle(t) + \left(\lim_{t_0\to t} \frac{\delta_{\ell\ell_0} - M_{\ell\ell_0}(t,t_0)}{t-t_0}\right)\langle C_{\ell_0}\rangle(t) = \frac{8\pi}{9} K |\mathbf{G}|^2 \delta_{\ell 1}\,,$$

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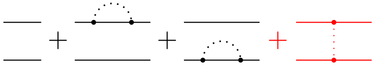
$$\langle U^A_{t,t_0} U^{B*}_{t,t_0} \rangle \simeq$$



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$$\frac{\mathrm{d}}{\mathrm{d}t} \langle C_\ell \rangle(t) + \left(\lim_{t_0 \to t} \frac{\delta_{\ell\ell_0} - M_{\ell\ell_0}(t, t_0)}{t - t_0} \right) \langle C_{\ell_0} \rangle(t) = \frac{8\pi}{9} K |\mathbf{G}|^2 \delta_{\ell 1} \,,$$

$\rightarrow$ Gradient source term is mixing into higher harmonics!

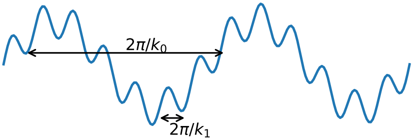
Toy model

- Isotropic turbulence tensor:

$$\langle \tilde{\omega}_i(\mathbf{k}) \, \tilde{\omega}_j^*(\mathbf{k}') \rangle = \frac{g(k)}{k^2} \left(\delta_{ij} - \hat{k}_i \hat{k}_j \right) \delta(\mathbf{k} - \mathbf{k}')$$

- Band-limited white noise:

$$g(k) = g_0 \quad \text{if} \quad k_0 \leq k < k_1$$

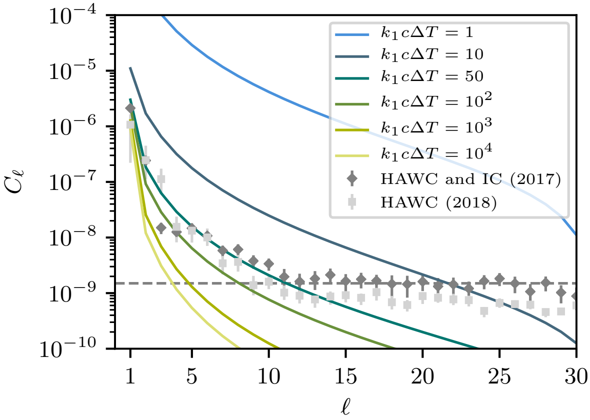


- In order to get local operators

$$\Delta T \equiv (t - t_0) \to 0 \quad \text{while} \quad k_1 \Delta T = \text{const.}$$

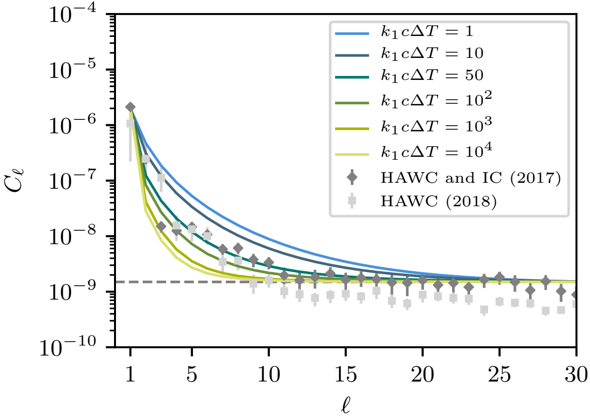
- Require $k_1 \Delta T > 1$ and $k_0 \Delta T \ll 1$

Results



- Fix source term $K|\mathbf{G}|^2$ to $10^{-4} k_0$
- No shot noise

Results



- Let source term $K|\mathbf{G}|^2$ float
- Add shot noise due to experimental statistics

Discussion

Good agreement over wide parameter range

→ Beware of cosmic variance:

$$\Delta C_\ell = \sqrt{2/(2\ell + 1)} \langle C_\ell \rangle$$

- The anisotropy of the ensemble average might not be perfectly dipolar
  Giacinti & Kirk, ApJ 835 (2017) 258
- Need to include regular field $\mathbf{\Omega}$
- Test different turbulence tensors

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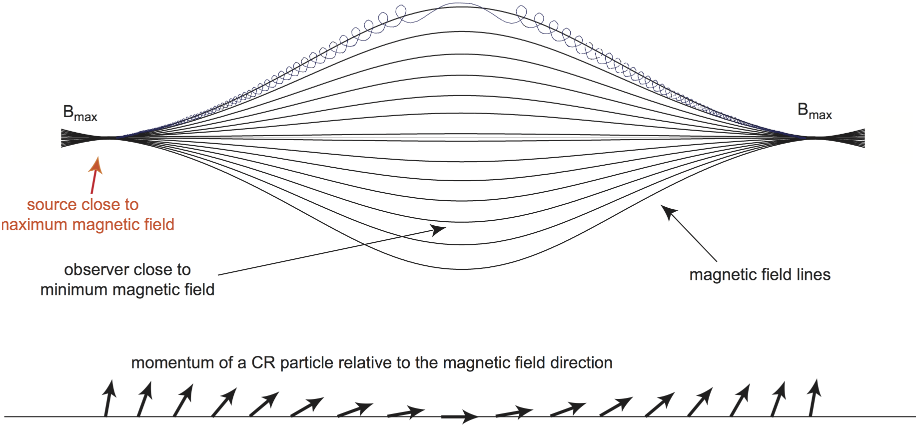
Focussing CRs

$$\frac{p_\perp^2}{2B} = \text{const.}$$



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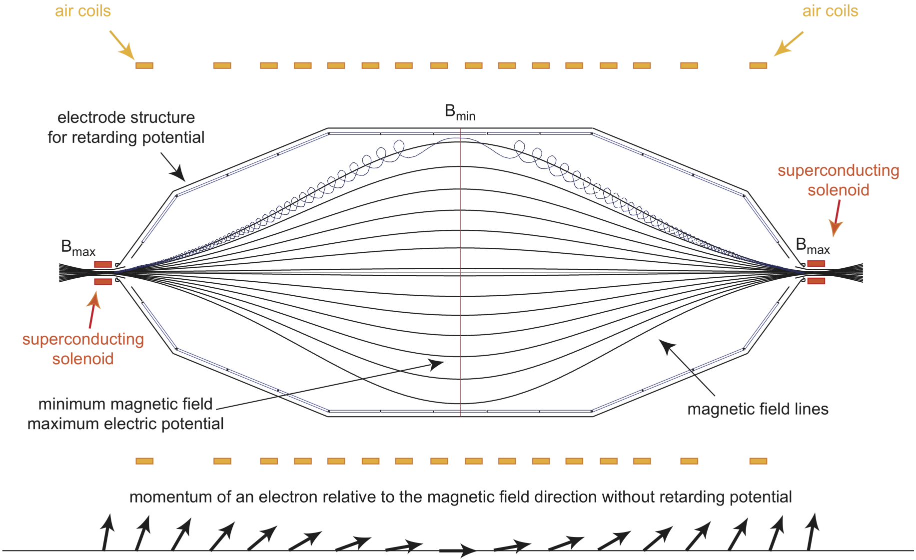

Beck *et al.*, JINST 9 (2014) P11020

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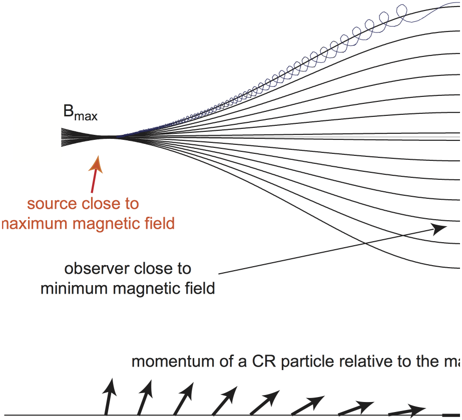


- beam width

$$\delta\theta \simeq \sqrt{\frac{B_{\text{min}}}{B_{\text{max}}}} \simeq 5^\circ \left(\frac{B_{\text{max}}/B_{\text{min}}}{100} \right)^{-1/2}$$

- beam can be subdominant

- source needs to be close to maximum → unnatural?
- small-scale turbulence will broaden beam → source needs to be closer than scattering length $\mathcal{O}(10)$ pc at 1 PeV

Non-uniform pitch-angle scattering

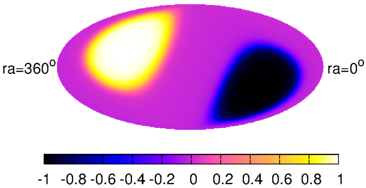
Solve Fokker-Planck equation but with $D_{\mu\mu} \neq D_0(1-\mu^2)$

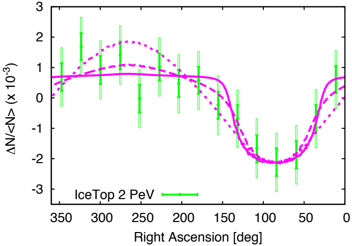
1. Goldreich-Sridhar turbulence $\rightarrow$ narrow peak in $D_{\mu\mu}$ $\rightarrow$ narrow beam in CRs

   Malkov *et al.*, ApJ 721 (2010) 750

2. modification of the large-scale anisotropy:
   - compute $D_{\mu\mu}$ in quasi-linear theory in various turbulence models
   - can have peak close to $\mu = 0$
   - consider higher-order terms in series in μ
   - large-scale anisotropy modified

   Giacinti & Kirk, ApJ 835 (2017) 258





Are heliospheric effects strong enough?

- Solar modulation in force field approximation with $\mathcal{O}(100)$ MeV potentials

$\rightarrow$ Can this effect TeV-PeV cosmic rays?

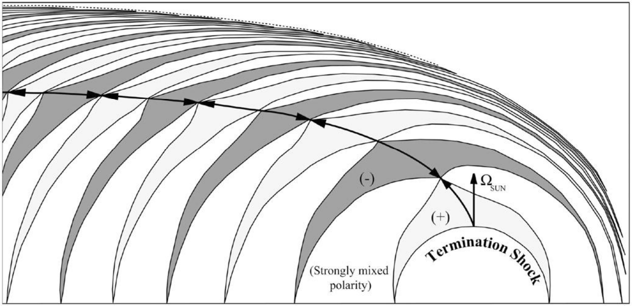
- Alignment of excess region with heliotail
- $r_g \simeq 200\,(R/\mathrm{TV})(B/\mu\mathrm{G})^{-1}\,\mathrm{AU}$ is $\lesssim$ size of heliosphere
- Need not modify isotropic flux, but only arrival directions:

Drury (2013)

- Electric field due to relative bulk speed of ISM CRs in heliosphere: $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$
- $v = 10\,\mathrm{km/s},\ B = 10\,\mu\mathrm{G} \rightarrow 1.5\,\mathrm{MV/AU}$
- If field coherent over 100 AU $\rightarrow$ 150 MV
- 10^{-4} effect for TeV particles

Explaining the excess in the heliotail

Lazarian and Desiati, ApJ 722 (2010) 188, Desiati and Lazarian, ApJ 762 (2013) 44



Nerney, Suess and Schmahl, JGR 100 (1995) 3463

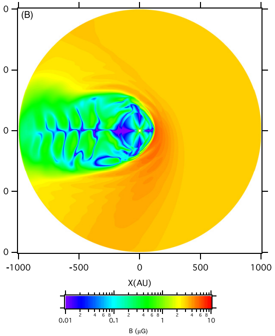
- Reconnection in the heliotail $\rightarrow$ harder spectrum in excess region
- Super-Alfvénic turbulence with $\lambda_{\mathrm{mfp}} \sim r_g$ $\rightarrow$ excess in the heliotail
- Misalignment of ISM flow and B direction $\rightarrow$ non-dipolar anisotropies
- Reconstruction errors of large-scale (angular) gradient $\rightarrow$ small-scale structure

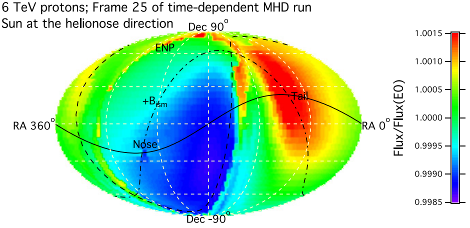
Detailed numerical model

Zhang, Zuo and Pogorelov, ApJ 790 (2014) 5

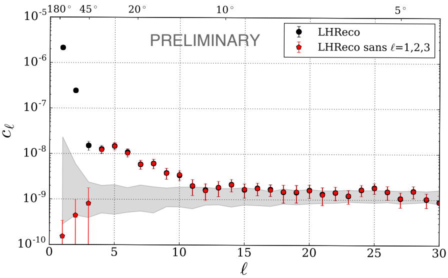
- state-of-the-art MHD model of heliosphere
- backtrack from initial distribution with $\nabla_\perp \ln n$, dipole and quadrupole

1. acceleration in electric fields
2. non-uniform pitch–angle scattering along the regular magnetic field
3. drift diffusion perpendicular to the field ("B–cross–gradient" forces)



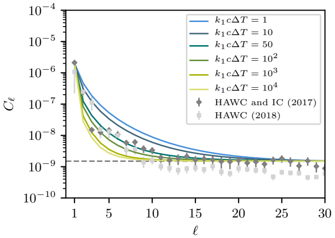


Summary



Observations

- anisotropies down to $\sim 5^\circ$
- power law in ℓ for $\ell > 5$
- no time-dependence



Small-scale turbulence model

Correlated propagation of particle pairs:

- Stochastic differential equation
- Diagrammatic technique
- Predicts power law spectrum

Time evolution operator

- Liouville's theorem:

$$\frac{\partial}{\partial t} f + \left(\mathcal{L}_0 + \delta\mathcal{L}(t)\right) f(t) = 0 \qquad\qquad \imath\hbar \frac{\partial}{\partial t} |\psi(t)\rangle - (H_0 + H_I) |\psi(t)\rangle = 0$$

- Formally solved as

$$f(\mathbf{r}, \mathbf{p}, t) = U_{t,t_0} f(\mathbf{r}, \mathbf{p}, t_0) \qquad\qquad |\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle$$

- With free propagator:

$$U^{(0)}_{t,t_0} = \exp\left[-\int_{t_0}^{t} \mathrm{d}t' \mathcal{L}_0(t')\right] \qquad\qquad U^{(0)}(t, t_0) = \exp\left[-\imath H_0 (t - t_0)/\hbar\right]$$

- And time evolution operator:

$$U_{t,t_0} = U^{(0)}_{t,t_0} \mathcal{T} \exp\left[-\int_{t_0}^{t} \mathrm{d}t' \underbrace{\left(U^{(0)}_{t',t_0}\right)^{-1} \delta\mathcal{L}(t') U^{(0)}_{t',t_0}}_{\sim\text{interaction picture Hamiltonian}}\right]$$

Mean Green's function

- Perturbative expansion (Dyson series):

$$U_{t,t_0} = U^{(0)}_{t,t_0} + \sum_{n\geq 1}(-1)^n \int_{t_0}^{t} \mathrm{d}t_n \int_{t_0}^{t_n} \mathrm{d}t_{n-1} \ldots \int_{t_0}^{t_2} \mathrm{d}t_1$$

$$\times\, U^{(0)}_{t,t_n} \delta\mathcal{L}(t_n) U^{(0)}_{t_n,t_{n-1}} \delta\mathcal{L}(t_{n-1}) \ldots \delta\mathcal{L}(t_1) U^{(0)}_{t_1,t_0} .$$

- But $\delta\mathcal{L}(t)$ is a random variable. So what is $\langle U_{t,t_0} \rangle$?
- Evaluate expectation values in Gaussian approximation:

$$\langle \delta\mathcal{L}(t_n)\delta\mathcal{L}(t_{n-1}) \ldots \delta\mathcal{L}(t_1) \rangle \simeq \langle \delta\mathcal{L}(t_n)\delta\mathcal{L}(t_{n-1}) \rangle \ldots \langle \delta\mathcal{L}(t_1)\delta\mathcal{L}(t_0) \rangle + \text{permut.}$$

Mean Green's function

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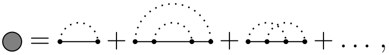
- Fourth order term:

$$\int_{t_0}^{t<t_4<t_3<t_2} \mathrm{d}t_4 \ldots \mathrm{d}t_1\, U^{(0)}_{t,t_4} \delta\mathcal{L}(t_4) U^{(0)}_{t_4,t_3} \delta\mathcal{L}(t_3) U^{(0)}_{t_3,t_2} \delta\mathcal{L}(t_2) U^{(0)}_{t_2,t_1} \delta\mathcal{L}(t_1) U^{(0)}_{t_1,t_0}$$
$$+ \int_{t_0}^{t<t_4<t_3<t_2} \mathrm{d}t_4 \ldots \mathrm{d}t_1\, U^{(0)}_{t,t_4} \delta\mathcal{L}(t_4) U^{(0)}_{t_4,t_3} \delta\mathcal{L}(t_3) U^{(0)}_{t_3,t_2} \delta\mathcal{L}(t_2) U^{(0)}_{t_2,t_1} \delta\mathcal{L}(t_1) U^{(0)}_{t_1,t_0}$$
$$+ \int_{t_0}^{t<t_4<t_3<t_2} \mathrm{d}t_4 \ldots \mathrm{d}t_1\, U^{(0)}_{t,t_4} \delta\mathcal{L}(t_4) U^{(0)}_{t_4,t_3} \delta\mathcal{L}(t_3) U^{(0)}_{t_3,t_2} \delta\mathcal{L}(t_2) U^{(0)}_{t_2,t_1} \delta\mathcal{L}(t_1) U^{(0)}_{t_1,t_0}$$

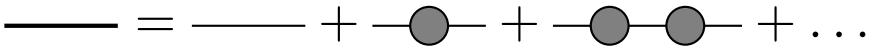
Resummation and Bourret approximation

- Full series convergent, but partial series can diverge

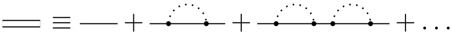
$\rightarrow$ Resummation of connected diagrams into "mass operator"



so summands in $\langle U_{t,t_0} \rangle$ factorise:



- Bourret approximation: approximate mass operator with its first term,



Diffusion on sphere

- For homogeneous and static turbulence and $\mathbf{\Omega} = 0$:

$$\langle U_{t,t_0} \rangle \simeq \mathrm{e}^{-\nu(t-t_0)\mathbf{L}^2}$$

- Diffusion equation in $\hat{\mathbf{n}}$:

$$\frac{\partial}{\partial t} f(t, \hat{\mathbf{n}}) - \nu \Delta f(t, \hat{\mathbf{n}}) = 0$$

- Laplacian on sphere (for $|\mathbf{r}| = 1$):

$$\Delta = -\mathbf{L}^2$$

- Solved by:

$$f(t, \hat{\mathbf{n}}) = \mathrm{e}^{-\nu(t-t_0)\mathbf{L}^2} f(t_0, \hat{\mathbf{n}}) = \langle U_{t,t_0} \rangle f(t_0, \hat{\mathbf{n}})$$

Bourret propagator describes isotropic pitch-angle scattering