Interferometry on the spherical sky

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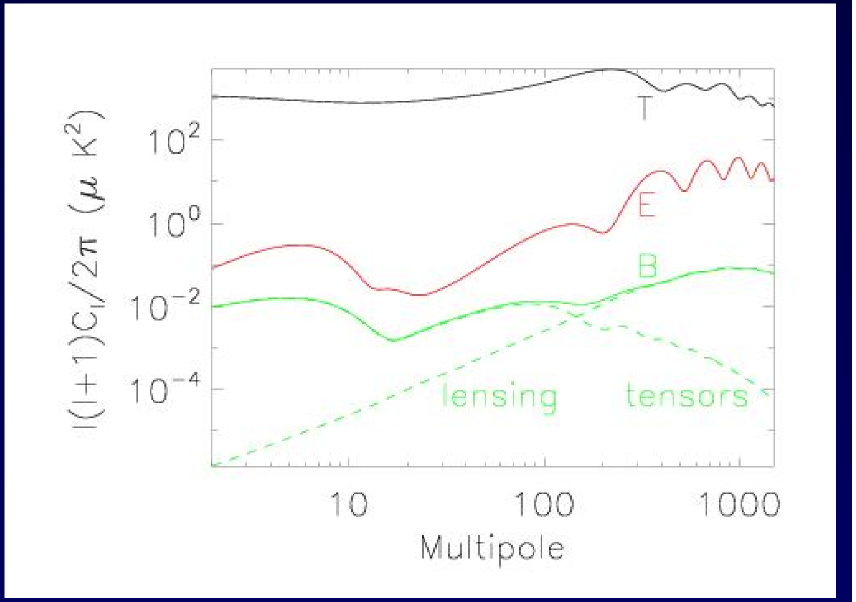
Beyond the flat sky

Large areas of sky needed for

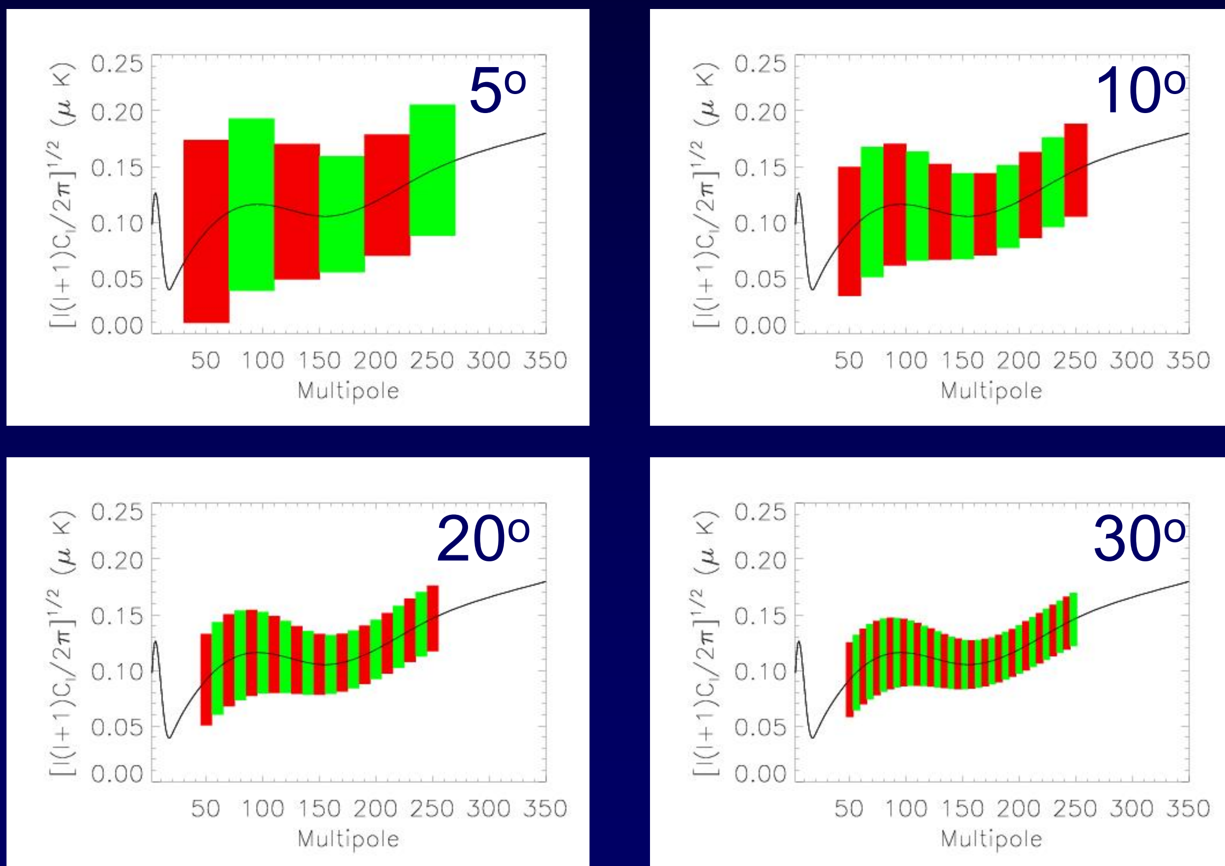
- Low l-values
  - » Reionization bump
- Fine l-space resolution

$$\Delta l \sim f_{\rm sky}^{-1/2}$$

  - » Structure in T, E, maybe even B.
  - » *Correlations between small patches far apart.*



Sky coverage matters



Ideal cosmic-variance-limited B mode experiment.

Small Δl requires keeping track of correlations over entire observed area.

Useful tools

- Simulating visibilities
  - » *Real space* → *Visibility space*
- Aperture synthesis
  - » *Visibility space* → *Real space*
- Likelihood analysis of power spectra
  - » $C_l \rightarrow \langle V_j V_k^* \rangle$

Visibilities from maps

Flat sky:

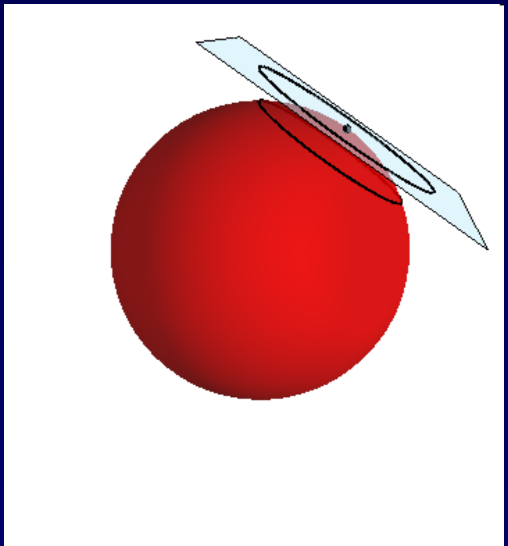
$$V(\vec{u};\vec{c}) = \int d^2\vec{x}\, A(\vec{x})T(\vec{x}-\vec{c})e^{2\pi i\vec{u}\cdot\vec{x}}$$

$$V(\vec{u};\vec{c}) = (\tilde{A}^* \star \tilde{T}_{\vec{c}})(\vec{u}) \qquad \tilde{T}_{\vec{c}}(\vec{q}) = \tilde{T}(\vec{q})e^{-2\pi i\vec{q}\cdot\vec{c}}$$

Convolution makes calculation of visibilities fast & easy.

Spherical sky:

Traditional approach: For a single pointing, just project onto the tangent plane.



Bad news:

1. $\tilde{T}$ not related in a simple way to a_{lm}.
2. Not natural to analyze multiple pointings on the same footing.

(Hobson & Maisinger 2002, Tegmark & Zaldarriaga 2008, …)

Learning to love the sphere

$$V(\vec{u};\mathcal{R}) = \int d^2\hat{n}\, A(\hat{n}) T(\mathcal{R}\hat{n}) e^{2\pi i \vec{u}\cdot\hat{n}}$$

$\mathcal{R}$ = rotation carrying center of antenna pattern to pointing center

$$V(\vec{u};\mathcal{R}) = \sum_{l,m,m'} D^l_{mm'}(\mathcal{R}) a_{lm'} F_{lm}(\vec{u}).$$

$$F_{lm}(\vec{u}) = \int d^2\hat{n}\, A(\hat{n}) Y_{lm}(\hat{n}) e^{2\pi i \vec{u}\cdot\hat{n}}.$$

(Ng 2001, McEwen & Scaife 2008)

For each u, visibilities for all $\mathcal{R}$ can be found by convolution:

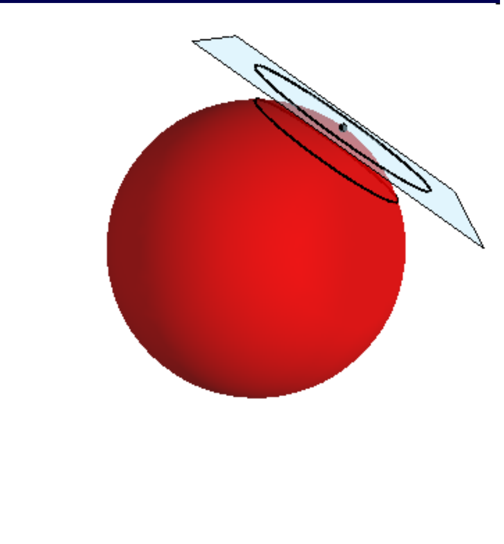
$$V(\vec{u};\mathcal{R}) = (A_u \star T)(\mathcal{R}), \qquad A_u(\hat{n}) = A(\hat{n}) e^{2\pi i \vec{u}\cdot\hat{n}}$$

$$\text{Time} \sim \mathcal{O}(l_{\max}^4 N_u)$$

(Wandelt & Gorski)

Aperture synthesis

- Flat-sky:
  - » Visibilities sample Fourier transform of (AT).
  - » As long as uv coverage is dense enough, you can just do an inverse F.T.
- Spherical sky:
  - » Analyze on the plane and project back, or do the analysis on the sphere.



Aperture synthesis on the sphere

$$\bar{T}(\hat{n}) \equiv T(\hat{n})A(\hat{n}) \equiv \sum_{l,m} \bar{a}_{lm} Y_{lm}(\hat{n}) \qquad V(\vec{u}) = 4\pi \sum_{l,m} i^l \bar{a}_{lm} j_l(2\pi u) Y_{lm}(\hat{u}).$$

- Given samples of V, can we solve for $\overline{a}_{lm}$?
- Easy cases:
  - » All baselines on a couple of spheres in *uvw* space are measured (not gonna happen).
  - » Densely sampled baselines in the *uv* plane.
- More realistic cases are equivalent to the map-making problem:

$$\vec{V} = \mathbf{A} \cdot \vec{a} \Rightarrow \hat{\vec{a}} = (\mathbf{A}^\dagger \cdot \mathbf{A})^{-1} \cdot \mathbf{A}^\dagger \cdot \vec{V} \qquad \mathrm{Time} \sim \mathcal{O}(f_{\mathrm{sky}}^3 l_{\mathrm{max}}^6)$$

- No particular advantage over projecting onto the plane.
- These are single-patch $\overline{a}_{lm}$, not full-sky a_{lm}.

(Kim 2007; McEwen & Scaife 2008)

Visibility covariance matrix

- To calculate likelihoods, need to know the theory covariance matrix window functions:

$$M_{jk} \equiv \langle V(\vec{u}_j, \mathcal{R}_j) V^*(\vec{u}_k, \mathcal{R}_k) \rangle \equiv \sum_l C_l W_l^{(jk)}$$

- When individual fields of view are large, do the exact calculation:

$$W_l^{(jk)} = \sum_{m,m'} F_{lm}(\vec{u}_j) D^l_{mm'}(\mathcal{R}_j \mathcal{R}_k^{-1}) F^*_{lm'}(\vec{u}_k)$$

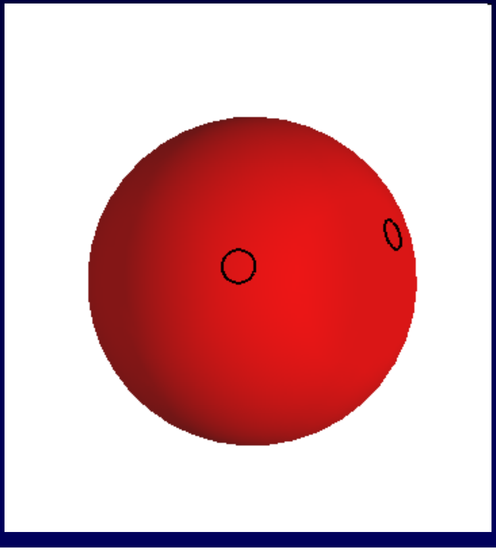
- Hardest case: Large FOV, large u (wide spacings).
  - » Need to go to high l.
  - » Flat-sky approximation no good.
  - » Not common in CMB experiments so far.
- There's a simpler way when FOV is small, even if pointing centers are widely separated.

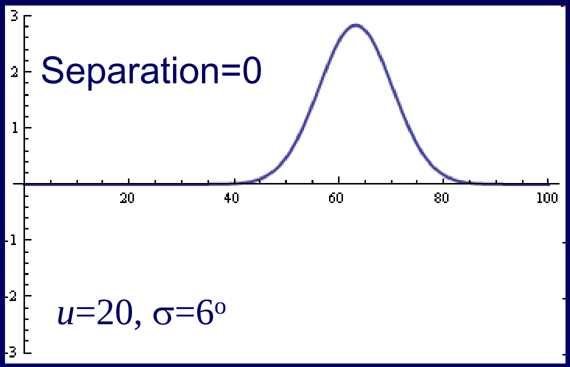
Small FOV, large separation

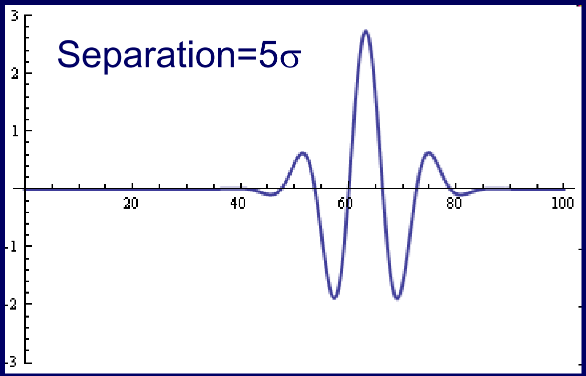
Question: Aren't the correlations very small in this case anyway?

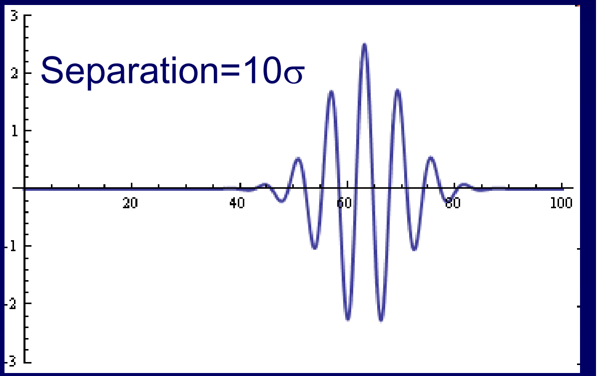
Answer 1: Yes, but they're exactly what you need to get good l-space resolution on the power spectrum.

Answer 2: And they're not always as small as you might think, particularly when both u's are about equal and parallel to the separation.



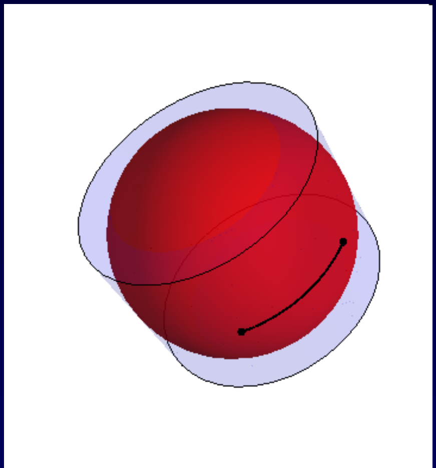






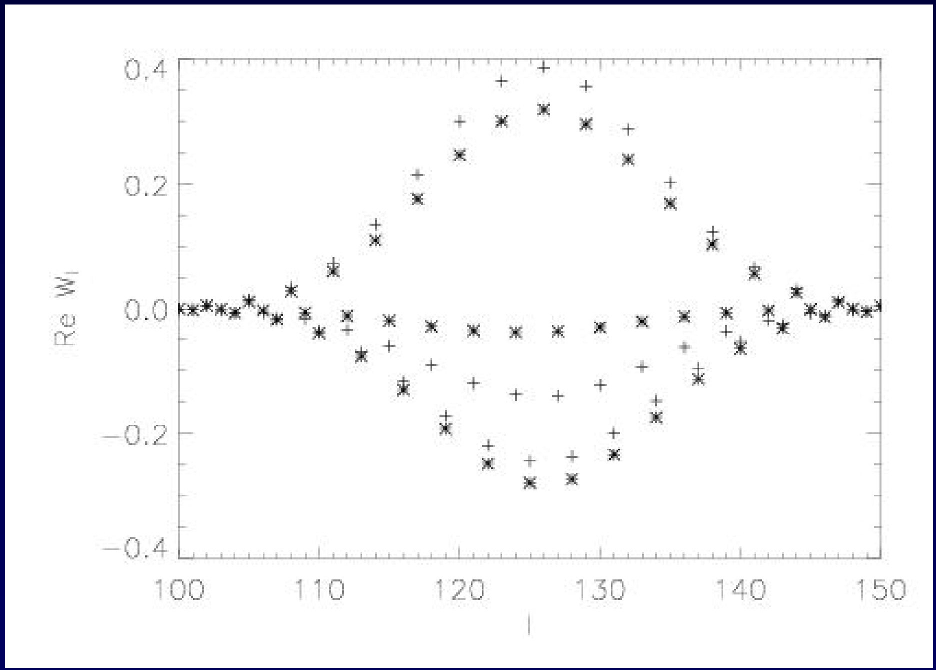
Cylindrical Approximation

- Each covariance matrix element depends only on values in the neighborhood of the two pointing centers.
- For small FOV, the sphere is well-approximated by a cylinder.
- Instead of spherical harmonic expansion, use Fourier transform on the cylinder (discrete / continuous).
- Cylindrical approximation is very accurate for beamsizes $< 20^{o}$, for arbitrarily large separation.



(Bunn & White 2007)

Window functions



Separation = 120°.

$\sigma = 5°$

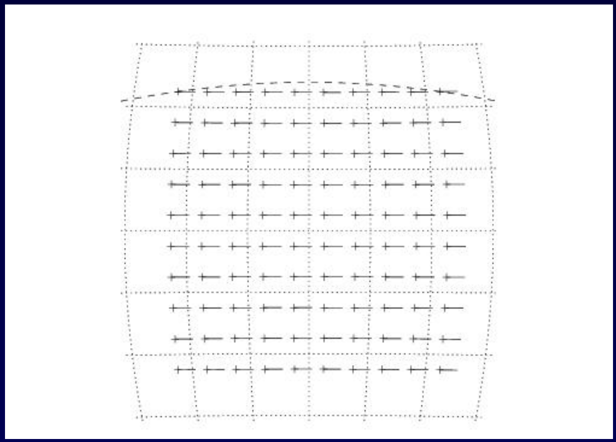
$u = 20$

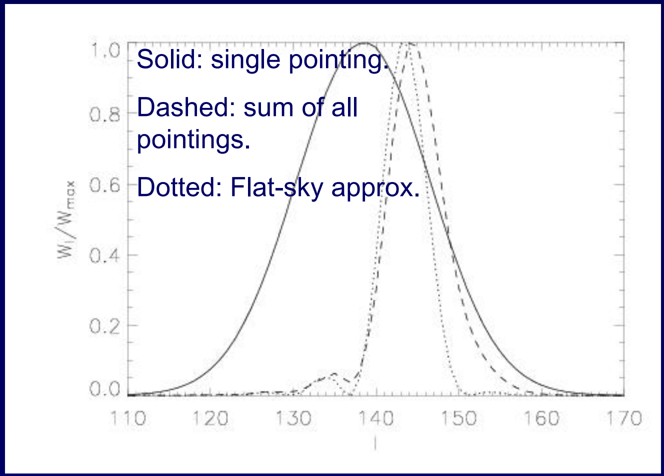
+: Cylindrical

*: Flat-sky

Example

- 10 x 10 grid with a single baseline for each pointing.
- All pointings separated by 5^o in θ,ϕ, with grid center on the equator.
- All baselines are identical in spherical coordinates:
- Baselines far from the equator aren't "really" parallel ➔ Flat-sky approximation overestimates correlation.

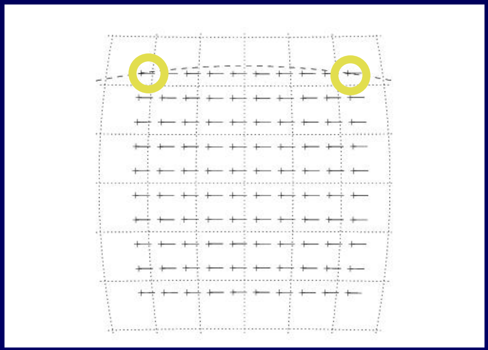




Polarization

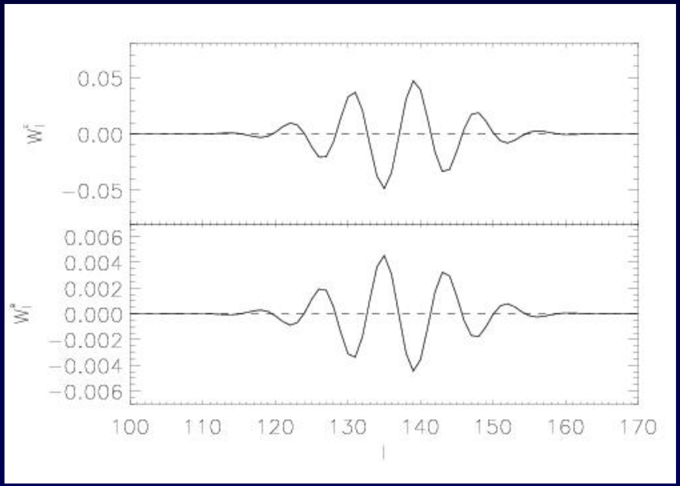
E and B window functions for Stokes Q for this pair of baselines.

Flat-sky overestimates correlations by ~200 in this case.

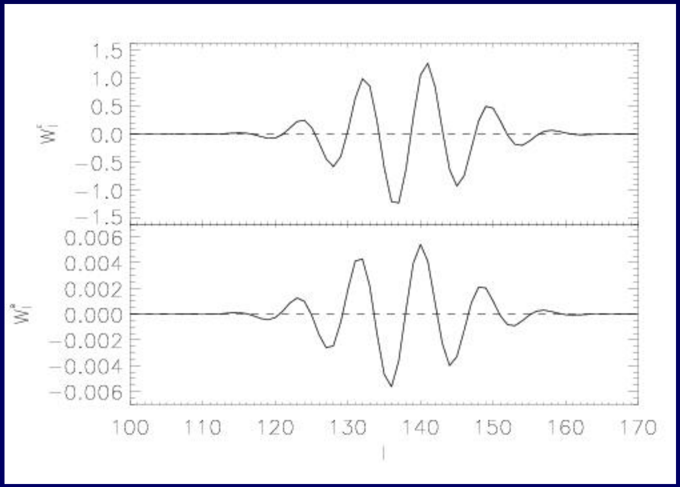


Cylindrical

(Exact calc. has negligible difference.)



Flat-sky



Conclusions

- Future experiments will need to go beyond the flat-sky approximation to get to low l and to improve l-space resolution.
- For some CMB work, it's more natural to work with data on the sphere, rather than projecting onto a tangent plane.
- At low l, exact expressions are computationally tractable (but messy).
- To get good resolution at high l, mosaics of small pointings are likely. Cylindrical approximation makes calculation of covariance window functions much easier in this regime.