Discrete Time and Frequency Wigner–Ville Distribution: Moyal's Formula and Aliasing

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***Abstract*—In this letter, we propose a new definition of the discrete time and frequency Wigner–Ville distribution. The proposed distribution not only displays a readable representation (small aliasing) but also exhibits unitarity and is easy to compute. We compare the time-frequency representation associated with this proposed definition with other existing ones.**

***Index Terms*—Time-frequency analysis.**

I. Motivation

Over the last 50 years, a large effort has been made to efficiently represent a signal jointly in time and frequency. This has led to a wide variety of possible time-frequency (TF) representations (TFRs) (see, e.g., [1]). The characteristics of the signal and the type of information of interest decide the most suitable choice of TFR to be considered for signal analysis.

The Wigner–Ville (WV) distribution is a common candidate among various quadratic TFRs, as it is simple and satisfies many interesting mathematical properties. For continuous time and frequency variables and a signal $x(t) \in L^2(\mathbb{R})$ or in the Hardy space $H^2(\mathbb{R})$, the WV distribution is defined as [2]

$$W_x(t,f) \equiv \int_{-\infty}^{+\infty} x(t+s/2)x^*(t-s/2)e^{-2\pi i f s}ds. \quad (1)$$

Besides other properties, the WV distribution obeys unitarity, a.k.a. Moyal's formula [1]

$$\iint_{-\infty}^{+\infty} W_x(t,f)W_y(t,f)\,dt\,df = \left|\int_{-\infty}^{+\infty} x(t)y^*(t)\,dt\right|^2. \quad (2)$$

As unitarity allows an equivalent TF formulation of operations performed either in time or in frequency, it is a central ingredient for the design of TF-based processing algorithms[1] [2].

The use of sampled signals demands unitary discrete time *and* frequency distributions (for evaluation with a computer). This extension from a continuous to a discrete TF plane is *not* straightforward. In the literature, several definitions of fully discrete and unitary WV have been proposed for periodic signals [4]–[6]. However, there is no satisfactory WV distribution associated with *band-limited nonperiodic* signals of *finite time support*. Available solutions either fail in obtaining unitarity or lead to a TFR complicated by many aliasing terms.

In Section II, we propose a fully discrete WV adapted to this case. It satisfies a set of useful properties listed in Table I (proofs are detailed in Section III) and displays a more "readable" representation (with aliasing terms of smaller amplitudes) than other definitions. In Section IV, we illustrate this last point and explain the origin of the residual aliasing with an example of linear chirp. Here, we give priority to the properties (primarily unitarity) as opposed to investigating aliased-free discrete WVs (see, e.g., [7]) with values *exactly* matching the continuous WV.

II. Proposal of a Discrete Time and Frequency WV

Let $x(k) \triangleq x(kt_s)$ with $k \in \mathcal{Z}_N \equiv \{0,1,\ldots,N-1\}$ sampling frequency f_s $(t_s = 1/f_s)$ be a discrete time series with an even number of samples N [with the best choice $N = 2^p$ for fast computation with fast Fourier transform (FFT)]. We define the discrete WV distribution of x for $(n,m) \in \mathcal{Z}_N \times \mathcal{Z}_{2N}$ as

$$w_x(n,m) \equiv t_s \sum_{k=-k_n}^{k_n} x(p_{n,k})x^*(q_{n,k})e^{-2\pi i mk/(2N)} \quad (3)$$

where[2] $k_n \equiv \min\{2n, 2N-1-2n\}$, $p_{n,k} \equiv \lfloor n+k/2 \rfloor$, $q_{n,k} \equiv \lfloor n-k/2 \rfloor$, and $\lfloor l \rfloor$ denotes the greatest integer less than or equal to l.

It is evident that $w_x(n,m)$ is a discrete Fourier transform (DFT) of an even sequence and, hence, is real. The time and frequency axes in physical units are $t_n = t_s n$ and $f_m = f_s m/(2N)$. Thus, the frequency axis gets sampled at twice the usual rate.

By separating the summation over even and odd k's in (3), it can be easily shown that $w_x(n,m)$ is a quadratic transform of $x(k)$, as defined in Table I, with kernel

$$H(n,m;j,k) = f_s e^{\pi i m(k-j)/N} \times [\delta(j+k-2n) + \delta(j+k-2n+1)] \quad (4)$$

where δ is the Kronecker symbol, i.e., $\delta(k) = 1$ if $k = 0$ and 0 otherwise. This kernel is the sum of two Kronecker deltas; the second delta can be obtained from the first by replacing $2n$ with $2n-1$. In Section IV, we will deduce from this relation a rule explaining the residual aliasing of this WV.

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[1]Moyal's formula provides the optimal TF statistic for the detection of chirp signals, a crucial result with applications in areas like gravitational wave detection [3].

[2]Equation (3) is formally equivalent to the definition B in [8, (73a)] provided a specific sampling of the frequency axis, i.e., $\theta = \pi m/N$. However, all results in [8] are obtained assuming a continuous frequency axis. The discretization of the frequency axis introduces significant difficulty in proving Moyal's formula.

TABLE I
LIST OF THE PROPERTIES OF THE PROPOSED WV DISTRIBUTION IN (3)

quadratic	$w_x(n,m) = t_s^2 \sum_{j,k=0}^{N-1} H(n,m;j,k)x(j)x^*(k)$
real	$w_x(n,m) = w_x^*(n,m)$
shift covariant	$y(n) = x(n-n_0)e^{2\pi i n m_0/N} \rightarrow w_y(n,m) = w_x(n-n_0, m-2m_0)$
time marginal	$\|x(n)\|^2 = f_s/(2N)\sum_{m=0}^{2N-1} w_x(n,m)$
frequency marginal	$\|X(m)\|^2 = t_s \sum_{n=0}^{N-1} w_x(n,2m)$
Moyal's formula	$\|t_s \sum_{n=0}^{N-1} x(n)y^*(n)\|^2 = t_s f_s/(2N)\sum_{n=0}^{N-1}\sum_{m=0}^{2N-1} w_x(n,m)w_y(n,m)$
Weyl's correspondence	$t_s \sum_{n=0}^{N-1}(\mathbf{H}x)(n)y^*(n) = t_s f_s/(2N)\sum_{n=0}^{N-1}\sum_{m=0}^{2N-1} L_{\mathbf{H}}(n,m)w_{xy}(n,m)$

III. PROPERTIES OF THE PROPOSED WV

In this section, we prove several properties of the fully discrete WV proposed in Section II.

A. Energy Distribution With Exact Marginals

We define the energy of the signal x by

$$e_x \equiv t_s \sum_{n=0}^{N-1} |x(n)|^2 = \frac{f_s}{N}\sum_{m=0}^{N-1}|X(m)|^2 \tag{5}$$

where the second half of the above equation is obtained from Parseval's theorem combined with the definition of the DFT $X(m) \equiv t_s \sum_{k=0}^{N-1} x(k)e^{-2\pi i k m/N}$. Thus, $|x(n)|^2$ and $|X(m)|^2$ can be interpreted as time and frequency energy distributions, respectively.

1) Time Marginal Property: For a fixed n, summing the WV in (3) marginally along the frequency axis, we have

$$\frac{f_s}{2N}\sum_{m=0}^{2N-1} w_x(n,m) = \sum_{k=-k_n}^{k_n} x(p_{n,k})x^*(q_{n,k})\delta_{2N}(k) \tag{6}$$

where $\delta_N(k) \equiv 1/N\sum_{m=0}^{N-1} e^{-2\pi i k m/N}$. Using the relation $\delta_N(k) = \delta([k]_N)$, where $[k]_N \equiv k - N\lfloor k/N \rfloor$, we obtain the time marginal property

$$\frac{f_s}{2N}\sum_{m=0}^{2N-1} w_x(n,m) = |x(n)|^2. \tag{7}$$

Summing both sides of (7) over n, we obtain the energy e_x. Thus, the proposed WV distribution is a TF energy distribution.

2) Frequency Marginal Property: We first note that the two Kronecker deltas in $H(n,m;j,k)$ divide into two parts the summation over j,k in w_x, as expressed in the first row of Table I. The first part contains terms with even $(j+k)$ and the second part with odd $(j+k)$. Therefore, we have

$$t_s \sum_{n=0}^{N-1} w_x(n,2m) = t_s^2 \sum_{\substack{j,k=0 \\ j+k \text{ even}}}^{N-1} e^{2\pi i m(k-j)/N} x(j)x^*(k) + t_s^2 \sum_{\substack{j,k=0 \\ j+k \text{ odd}}}^{N-1} e^{2\pi i m(k-j)/N} x(j)x^*(k). \tag{8}$$

In other words, the right-hand side of (8) gives all possible combinations of $j,k \in \mathcal{Z}_N$, yielding the frequency marginal property, as stated in Table I.

B. Covariance to Time and Frequency Shifts

Let $y(k) = x(k-n_0)\exp(2\pi i m_0 k/N)$ be the TF shifted version of $x(k)$ with n_0 and $m_0 > 0$ (the proof is similar for $n_0, m_0 < 0$). In physical units, the time and frequency shifts are $\Delta_t = t_s n_0$ and $\Delta_f = f_s m_0/N$. Since the signal $x(k)$ is of finite support $\mathcal{Z}_N$ and is bandlimited, the time and frequency shifts in the signal give rise to boundary effects. To ensure that the shifted signal $y(k)$ still fully lies in the TF plane, we assume $x(k) = 0$, for $k \notin \mathcal{Z}_N$ and $k \in \{N-n_0, \ldots, N-1\}$. Let the time axis be oriented from left to right. As $x(k)$ is translated to the right, only zeros enter from the left and zeros exit to the right of $\mathcal{Z}_N$. Further, $x(k)$ is assumed to be supported in frequency up to $f_s/2 - \Delta_f$. In order to prove the covariance of the proposed WV under time and frequency shifts, we need to obtain the relation between w_x and w_y. We have

$$w_y(n,m) = t_s \sum_{k=-k_n}^{k_n} x(p_{n',k})\, x^*(q_{n',k})e^{-2\pi i m' k/(2N)} \tag{9}$$

where $n' \equiv n - n_0$ and $m' \equiv m - 2m_0$. For $0 \le n \le (n_0 - 1)$, all values of $w_y(n,m)$ are zero. For $n_0 \le n \le (N/2 - 1 + n_0)$, by definition, $k_{n'} = 2n - 2n_0 < k_n$. The product $x(p_{n',k})\, x^*(q_{n',k})$ vanishes for $\pm k \in \{k_{n'}+1, \ldots, k_n\}$ and can be suppressed from the sum. For $(N/2 + n_0) \le n \le (N-1)$, $k_{n'} = k_n + 2n_0$. The product terms $x(p_{n',k})\, x^*(q_{n',k})$ being zeros for $\pm k \in \{k_n + 1, \ldots, k_{n'}\}$ can be added to the sum.

Thus, we revise the limits of summation and replace k_n by $k_{n'}$ in (9). We conclude that $w_y(n,m) = w_x(n',m')$. This implies that w_y is obtained by translating w_x in time and frequency with $(n,m) \mapsto (n - n_0, m - 2m_0)$ or equivalently $(t_n, f_m) \mapsto (t_n - \Delta_t, f_m - \Delta_f)$ in physical units.

C. Moyal's Formula

Let w_x and w_y be the WV distributions of two discrete time series $x(k)$ and $y(k), k \in \mathcal{Z}_N$. From the left-hand side of the discretized version of Moyal's formula and equation (3), we get

$$\begin{aligned} t_s \frac{f_s}{2N} \sum_{n=0}^{N-1}\sum_{m=0}^{2N-1} w_x(n,m)w_y(n,m) &= t_s^2 \sum_{n=0}^{N-1}\sum_{k,k'=-k_n}^{k_n} \delta_{2N}(k+k')x(p_{n,k})x^*(q_{n,k}) \\ &\quad \times y(p_{n,k'})y^*(q_{n,k'}). \end{aligned} \tag{10}$$

Note that the maximum value of k_n is $N-1$, $(k+k')$ takes all possible values between $\pm(2N-2)$. Thus, $\delta_{2N}(k+k')$ is nonzero only when $k = -k'$, yielding

$$\begin{aligned} t_s \frac{f_s}{2N} \sum_{n=0}^{N-1}\sum_{m=0}^{2N-1} w_x(n,m)\, w_y(n,m) &= t_s^2 \sum_{n=0}^{N-1}\sum_{k=-k_n}^{k_n} x(p_{n,k})x^*(q_{n,k})y^*(p_{n,k})y(q_{n,k}). \end{aligned} \tag{11}$$

A step toward the proof of Moyal's formula is to make the change of running variables, i.e., to apply the mapping $\mathcal{M} : (n,k) \mapsto (p,q) \equiv (p_{n,k}, q_{n,k})$. The domain D of the mapping $\mathcal{M}$ is restricted by the summation limits in the right-hand side

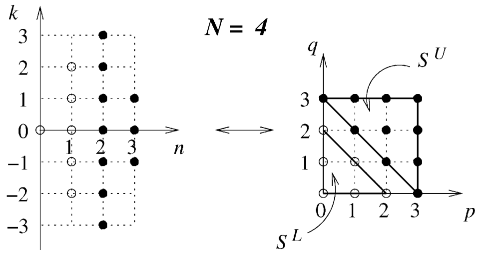

Fig. 1. Mapping $\mathcal{M} : (n,k) \mapsto (p,q)$.

of (11), which is illustrated for $N = 4$ in Fig. 1. Since $0 \leq n \pm k/2 < N$, the image of this mapping is a subset of the $N \times N$ discrete square S in the $p-q$ plane, i.e., $\mathcal{M}(D) \subset S \equiv \mathcal{Z}_N \times \mathcal{Z}_N$.

The geometrical interpretation of this mapping gives clear understanding of its nature. For even $k \equiv 2l$, we have $p+q = (n+l)+(n-l) = 2n$, whereas for odd $k \equiv 2l+1$, we get $p+q = (n+l)+(n-l-1) = 2n-1$. Thus, for a given $n \in \mathcal{Z}_N$, the set of ordered pairs (n,k), when k runs from $-k_n$ to k_n, gets mapped to two secondary diagonals of S, i.e., all terms having $p+q=2n$ and $p+q = 2n-1$.

We further elaborate this by splitting the summation over n in the right-hand side of (11), in turn splitting domain D into two parts. The first $N(N-1)/2$ terms of the domain D (for $0 \leq n \leq N/2-1$) map to lower half of the triangle S^L of S, and the remaining $N(N+1)/2$ terms (for $N/2 \leq n \leq N-1$) get mapped to the upper half triangle S^U of S (see Fig. 1).

It is evident that the two distinct pairs in D are transformed by $\mathcal{M}$ into two distinct images in $\mathcal{M}(D)$. A direct calculation shows that the cardinal card$(D) = \sum_{n=0}^{N-1} 2k_n + 1 = N^2$ is equal to card(S). This proves $\mathcal{M}$ to be one to one from D to S. Thus, we make the change of variables from dependent $(n,k) \in D$ to independent $(p,q) \in S$, which proves the unitarity of WV

$$t_s \frac{f_s}{2N} \sum_{n=0}^{N-1} \sum_{m=0}^{2N-1} w_x(n,m) w_y(n,m) = t_s^2 \sum_{p,q=0}^{N-1} x(p)x^*(q)y^*(p)y(q) = \left| t_s \sum_{n=0}^{N-1} x(n)y^*(n) \right|^2. \tag{12}$$

D. Weyl's Correspondence

Let $\mathbf{H}$ be the linear operator with kernel $h(k,k')$ acting on $x(k)$ for $k,k' \in \mathcal{Z}_N$ such that $(\mathbf{H}x)(k) = t_s \sum_{k'=0}^{N-1} h(k',k)x(k')$. We define *Weyl's symbol* $L_{\mathbf{H}}$, corresponding to $\mathbf{H}$, as

$$L_{\mathbf{H}}(n,m) \equiv t_s \sum_{k=-k_n}^{k_n} h(p_{n,k}, q_{n,k}) e^{-2\pi i mk/(2N)}. \tag{13}$$

We show that Weyl's symbol allows one to reformulate quadratic forms involving $\mathbf{H}$ in the TF plane through Weyl's correspondence. Let $w_{xy}(n,m)$ be the cross WV distribution obtained from (3) by replacing the second term in x by y. From (13) and using the mapping in Section III-C, we get Weyl's correspondence

$$t_s \frac{f_s}{2N} \sum_{n=0}^{N-1} \sum_{m=0}^{2N-1} L_{\mathbf{H}}(n,m) w_{xy}(n,m) = t_s^2 \sum_{p,q=0}^{N-1} h(p,q)x(p)y^*(q) = t_s \sum_{n=0}^{N-1} (\mathbf{H}x)(n) y^*(n). \tag{14}$$

IV. Implementation and Examples

Let $\tilde{x}(2k) = \tilde{x}(2k+1) = x(k), \forall k \in \mathcal{Z}_N$ be the time series of size $2N$ such that each sample of $x(k)$ is duplicated. Noting that $\tilde{x}(k) = x(\lfloor k/2 \rfloor)$, the definition in (3) can be rewritten as

$$w_x(n,m) = t_s \sum_{k=-k_n}^{k_n} \tilde{x}(2n+k)\tilde{x}^*(2n-k) e^{-2\pi i mk/(2N)}. \tag{15}$$

Therefore, the implementation[3] of w_x involves computation of N FFTs with time base $2N$ of the local autocorrelation function of $\tilde{x}(k)$ at even locations. The overall cost scales with $O(N^2 \log N)$ and the output can be stored in a $2N \times N$ array.

A. Three Variants of the Fully Discrete WV Distribution

We review several alternative definitions of WV with a special emphasis on Moyal's formula.

Claasen and Mecklenbräuker (CM): The discretization of (1) (first published in [9]) requires $x(k)$ to be replaced by an equivalent *halfband signal*, i.e., signal whose Fourier transform is supported over half the Nyquist band. The halfband counterpart of a signal sampled at the Nyquist rate can be computed in various ways. We choose the sequence $\hat{x}(k)$, which results from the resampling of $x(k)$ at twice its rate. This yields a discrete WV similar to (3) (same k_n)

$$w_x^{(0)}(n,m) \equiv t_s \sum_{k=-k_n}^{k_n} \hat{x}(2n+k)\hat{x}^*(2n-k) e^{-2\pi i mk/(2N)} \tag{16}$$

with the difference that the oversampled signal $\hat{x}$ is used in place of the signal $\tilde{x}$ with duplicated samples. The sampling of the TF plane is the same as before, i.e., an array of size $2N \times N$. The representation given by (16) is closely related to the continuous one, i.e., $w_x^{(0)}(n,m) = W_x(t_n, f_m)$. However, it does *not* satisfy Moyal's formula.

Peyrin and Prost (PP): The discretization of (1) presented in [5] leads to

$$w_x^{(1)}(n,m) \equiv t_s \sum_{k=0}^{N-1} x(k)x^*([n-k]_N) e^{-2\pi i m(k-n/2)/N}. \tag{17}$$

[3]Freely distributed scripts are available at http://www.obs-nice.fr/ecm for reproducing all the illustrations presented here.

PP can be stored in a $2N \times 2N$ array. It does not use a halfband signal and, thus, gets affected by aliasing. However, $w_x^{(1)}(n,m)$ is a TF energy distribution and satisfies Moyal's formula

$$\frac{t_s}{2}\frac{f_s}{2N}\sum_{n,m=0}^{2N-1} w_x^{(1)}(n,m)w_y^{(1)}(n,m) = \left|t_s \sum_{n=0}^{N-1} x(n)y^*(n)\right|^2. \tag{18}$$

Richman et al. and O'Neill et al. (RO): Two approaches—one based on group theoretic arguments [6] and the other on an axiomatic method [4]—independently yielded

$$w_x^{(2)}(n,m) \equiv t_s \sum_{k=0}^{N-1} x([n+2^{-1}k]_N) \times x^*([n-2^{-1}k]_N)e^{-2\pi imk/N} \tag{19}$$

where[4] 2^{-1} denotes an integer $n \in \mathcal{Z}_N$ such that $[2n]_N = 1$. It exists for odd $N > 2$, and thus, $2^{-1} = (N+1)/2$. RO can be stored in an array of size $N \times N$ and satisfies Moyal's formula

$$t_s \frac{f_s}{N}\sum_{n,m=0}^{N-1} w_x^{(2)}(n,m)w_y^{(2)}(n,m) = \left|t_s \sum_{n=0}^{N-1} x(n)y^*(n)\right|^2. \tag{20}$$

B. Comparisons and Aliasing Issue Via a Simulation Example

Here, we compare the proposed WV with the other WVs reviewed in Section IV-A. Further, we explain aliasing in this proposed WV. For both, we use the example of a linear chirp, which is a frequency modulated signal $x(k) = \exp(i\phi(k))$ with phase $\phi(k) = \pi k^2/(2(N-1))$ and frequency $f(k) \equiv (2\pi)^{-1}\dot{\phi}(k) = k/(2(N-1))$ linearly increasing from 0 to $1/2$ over $\mathcal{Z}_N$.

Comparison With Other TFRs: In Fig. 2, we present various TFRs; CM in (a), PP in (b), RO in (c), and the proposed WV in (d) for the linear chirp with $f_s = 1$ and $N = 128$ for (a), (b), and (d), while $N = 127$ for (c). The proposed WV in (d) is less affected by aliasing terms[5] compared to (b) and (c). It is closer to what one expects from continuous TF analysis [see (a)].

Understanding Aliasing in the Proposed WV: Recall that the kernel $H(n,m;j,k)$ is composed of two Kronecker deltas. Thus, the proposed WV can be rewritten as the sum of two terms: $w_x(n,m) = v(n,m) + v'(n,m)$. The first term has a periodicity of N, i.e., $v(n,m) = v(n,m+N)$. The second term can be obtained from the first by a transformation $v'(n,m) = v(n-1/2,m)$, a small time shift operation that consequently makes a periodic sign change. Thus, the second term has a periodicity of $2N$ and switches sign at half its period, i.e., $v'(n,m) = -v'(n,m+N) = v'(n,m+2N)$. Aliasing can be understood by comparing these two terms.

We demonstrate this with the above-described linear chirp. A closed-form expression for the WV of the above linear chirp can be obtained using $v(n,m) = \sin(\psi(n,m)\kappa_n)/\sin\psi(n,m)$ with $\psi(n,m) = 2\pi(f(n) - m/(2N))$ and $\kappa_n = \min\{2n+1, 2N-2n-1\}$. The two terms $v(n,m)$ and $v'(n,m)$ [see Fig. 2(e) and (f)] add constructively in the positive frequencies $(0 \le m \le N-1)$, whereas due to the small time shift between them, their destructive addition in the negative frequencies leaves behind the residual aliasing in WV [see Fig. 2(d); $N \le m \le 2N-1$]. This rule for the construction of aliasing terms in the proposed WV is not restricted to this example and can be applied in general. In fact, it is an artifact of the kernel H as noted in Section II.

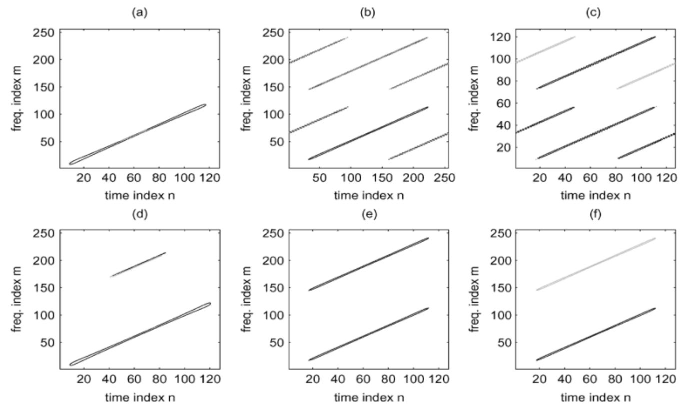

Fig. 2. *Linear chirp*—Top: various TFRs defined in Section 4: (a) CM, (16), (b) PP, (17), and (c) RO, (19). Bottom: (d) proposed WV in (3), (e) first term $v(n,m)$, and (f) second term $v'(n,m)$. We show two contours at levels $+e_x/4$ (black) and $-e_x/4$ (gray).

V. Concluding Remarks

In this letter, we have proposed a new fully discrete WV distribution. We have shown that it not only satisfies most of the important properties of its continuous counterpart but also displays a representation close to the continuous case (with small aliasing). In summary, the proposed distribution can prove to be an important tool for TF-based signal processing.

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[4]Note that (17) and (19) can be equivalently formulated by removing the modulo $[\cdot]_N$ and assuming $x(k)$ to be N-periodic.

[5]We refer to "aliasing terms" as artifacts appearing in the discrete WV but not in the continuous one. These terms can be produced by time and/or frequency folding of signal components, e.g., when the local autocorrelation function is undersampled. They can also result from the quadratic interaction (interference) of the signal components with their periodic copies.