Indirect method of integration: towards orbital evolution for EMRIs

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Plan of the talk

- Perturbations in RW gauge, $\Psi$ properties and jump conditions
- Indirect method: application for radial and generic orbits
- Orbital evolution: 4\textsuperscript{th} order, limits of the RW gauge, constraints and perspectives, comparison of approaches
Perturbations in RW gauge
Indirect method
Orbital evolution

The wave equation

- Perturbation of the Einstein Equation at first order
  \[ g_{\mu\nu} = g_{\mu\nu}^{\text{Schwarzschild-Droste}} + h_{\mu\nu} + \mathcal{O}(h^2) \]

- Regge-Wheeler-Zerilli equation for each \((l,m)\)-mode
  \[
  \left[ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^*} - V_\ell(r) \right] \psi_{\ell m}(t, r) = S_{\ell m}(t, r)
  \]
  \(r^* = r + 2M \ln \left( \frac{r}{2M} - 1 \right)\) is the tortoise coordinate.

- RW gauge \(\psi \in C^{-1}\) (discontinuous), \(h_{\mu\nu} \in C^0\) for radial infall
  \(h_{\mu\nu} \in C^{-1}\) for generic orbits.
Perturbations in RW gauge
Indirect method
Orbital evolution

\[ \Psi \text{ properties} \]

\[ \left[ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^*^2} - V(r) \right] \Psi(t, r) = F(t, z_g(t))\delta' + G(t, z_g(t))\delta \]

From the visual inspection of the RWZ wave equation, \( \Psi \in C^{-1} \) continuity class

\[ \Psi(t, r) = \Psi^+(t, r) \Theta_1 + \Psi^-(t, r) \Theta_2 \]

Jump condition on the wave function

\[ [\Psi]_{z_g} = \Psi^+(t, z_g(t)) - \Psi^-(t, z_g(t)) \]

where \( \Psi^\pm(t, z_g(t)) = \lim_{r \to z_g^\pm} \Psi(t, r) \)

\( \Theta_1 = \Theta(r - z_g), \Theta_2 = \Theta(z_g - r) \) Heaviside step distributions

\( \delta = \delta(r - z_g), \delta' = \frac{\partial}{\partial r}\delta(r - z_g) \) Dirac distribution and its spatial derivative

\( z_g = z_g(t) \) trajectory of the particle

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Jump conditions

\[
[\Psi]_{\text{zg}} = \frac{F(\text{zg})}{f(\text{zg})^2 - \dot{\text{zg}}^2}
\]

\[
[\partial_r \Psi]_{\text{zg}} = \frac{1}{f(\text{zg})^2 - \dot{\text{zg}}^2} \left[ G(\text{zg}) + \left( f(\text{zg}) \frac{df}{dr} (\text{zg}) - \ddot{\text{zg}} \right) [\Psi]_{\text{zg}} - 2\dot{\text{zg}} \frac{d}{dt} [\Psi]_{\text{zg}} \right]
\]

\[
[\partial_t \Psi]_{\text{zg}} = \frac{d}{dt} [\Psi]_{\text{zg}} - \dot{\text{zg}} [\partial_r \Psi]_{\text{zg}}
\]

where \( f(r) = (1 - 2M/r) \)
Indirect method I

- Integration domain discretised by 2-dimensional uniform mesh \((r^*, t)\).

- Empty cells (Lousto and Price, Martel and Poisson, Martel, Lousto, Haas)

\[
\psi_A^{\ell m} = -\psi_D^{\ell m} + \left( \psi_B^{\ell m} + \psi_C^{\ell m} \right) \left( 1 - \frac{\Delta r^*}{2} V(r) \right) + \mathcal{O}(\Delta r^{*2})
\]
Indirect method II

- The forward time value at the upper node of the \((r^*, t)\) grid cell is exclusively obtained by
  i) the preceding node values in the past light cone,
  ii) analytic expressions from the jump conditions on \(\Psi\) and its derivatives.
There is no direct integration of the singular source term or direct dealing of the potential.

\[
\Psi_{\ell m}^A = b \Psi_{\ell m}^B + c \Psi_{\ell m}^C + d \Psi_{\ell m}^D + [\Psi_{\ell m}]_\sigma (a - c + d) + [\partial_t \Psi_{\ell m}]_\sigma \Delta r^* (a - d) +
\]

\[
[\partial_{r^*} \Psi_{\ell m}]_\sigma (\epsilon (a - c + d) - \Delta r^* (a + d)) + O(\Delta r^{*2})
\]

with \(\Delta r^* = ||BC||/2, \quad \epsilon = ||\sigma C||\) and \(a, b, c, d\) are constant depending on the way the particle crosses the cell.
Indirect method III: results

Radial fall from $r_0 = 5(2M)$ for the mode $\ell = 2$.

Odd parity waveform for an eccentric orbit $(e, p) = (0.5, 7.2)$ for the mode $(\ell, m) = (2, 1)$.

Even parity zoom-whirl waveform orbit $(e, p) = (1.0, 8.001)$ for the mode $(\ell, m) = (2, 2)$. 
The indirect method may be extended to higher orders for any orbit, for increased accuracy but especially for computing back-action (SF effects).

Back-action determination implies the computation of perturbation derivatives (third derivative of $\psi$) and thus a $4^{th}$ order scheme.

$$\Psi_A^\ell = \sum_i q_i \Psi_i^\ell + \sum_{n+m<4} \tilde{q}_i t_i^{(n,m)} \left[ \partial_{r^*} \partial_t^m \Psi^\ell \right]_\sigma + O(\Delta r^*_5)$$

$q_i$ is constant and depending on the way the trajectory crosses the stencil, $\tilde{q}_i = 0$ if $r^*_i < z_g^*(t_i)$, $\tilde{q}_i = q_i$ else.

$i = \{B, C \ldots J\}$
Perturbations in radial fall

$4^{th}$ order accuracy gives access to the perturbation metric $h_{\mu\nu} = \begin{pmatrix} f H_2 & H_1 \\ H_1 & f^{-1} H_2 \end{pmatrix}$
Preliminary conclusions and way forward

- The indirect method is a powerful analytical tool applicable to any orbit.

- Our aim is the computation of orbital evolution.
  1. In RW gauge, the discontinuity of the perturbations doesn’t allow the computation of SF (except for radial fall, where there is a regular inter-gauge transformation + Riemann-Hurwitz $\zeta$ function).
  2. The self-consistent evolution "a la" Gralla-Wald is conceivable only in the harmonic gauge (where the SF is defined). **Though:**
  3a. An **iterative** evolution of the back-action may be conceived in other gauges.
  3b. The RWZ source term is written in terms of $R$, $\Phi$, $\Theta$ and time derivatives and it can accommodate **non-geodesic** orbits.
  4. Advanced and parallel computing is envisaged.

References
Fundamentals: motion deviation from background geodesic

We define two geodesics: one in the background -b- (unperturbed but curved) geometry, the other in the full -f- geometry

\[
\frac{D^2 z^\alpha}{d\tau^2} = \frac{d^2 z^\alpha}{d\tau^2} + b\Gamma^\alpha_{\mu\nu} u^\mu u^\nu = \frac{d^2 \hat{z}^\alpha}{d\tau^2} + b\Gamma^\alpha_{\mu\nu} \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} = 0
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\(\tau\) and \(\lambda\): proper time in the -b- and -f- metric, respectively.

\(\hat{z}^\alpha = z_g^\alpha + \Delta z^\alpha\): coordinates of \(m\) in \(g_{\mu\nu} + h^*_{\mu\nu}\), where \(h^*_{\mu\nu}\) is the DeWh radiative (effective) or the MiSaTuQuWa tail perturbation.

Subtracting the two geodesics, after some manipulation

\[
\frac{D^2 \Delta z^\alpha}{d\tau^2} = -\underbrace{R^\alpha_{\beta\mu\nu} u^\mu \Delta z^\beta u^\nu}_{\text{Background geodesic deviation}} - \frac{1}{2} \underbrace{(g^{\alpha\beta} + u^\alpha u^\beta)(2h^*_{\mu\beta;\nu} - h^*_{\mu\nu;\beta})u^\mu u^\nu}_{\text{Self – acceleration}}
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\text{Self – acceleration}
\]

What about evolution?
Evolution guidelines and constraints for BH parameters

The classic (1913+16 binary pulsar) way to take into account radiation reaction is by energy balance:

\[ E_{\text{rad}} = \Delta E_{\text{orb}}. \]

Energy conservation cannot be imposed at a given instant, but only adiabatically, exempli gratia averaging over a period, when the radiation reaction time scale is larger than the orbital period.

An adiabatic evolution impedes to calculate the conservative forces, id est contributions to the self-force which are not associated with energy or angular momentum loss.

The 'self-force Capra programme' aims at the computation in non-adiabatic situations. Three directions are investigated: a) tailored arrangements; b) 2nd order self-force formalism; 3) self-consistent "a la Gralla-Wald.

GW prescription 1: At later times any perturbative approach of whatever order is destined to fail of a certain amount: it is preferable to construct the trajectory by continuously correcting (at 1st order) the position of the captured body.

2: if continuous, the correction doesn't need the BGD term.

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Correspondence: SF ($\tau$) and pragmatic ($t$) approaches

The SF way

\[
\frac{D^2 \Delta z^\alpha}{d\tau^2} = -R_{\mu\beta\nu}^\alpha u^\mu \Delta z^\beta u^\nu
\]

Background geodesic deviation

\[-\frac{1}{2} (g^{\alpha\beta} + u^\alpha u^\beta)(2h^*_{\mu\beta;\nu} - h^*_{\mu\nu;\beta})u^\mu u^\nu\]

Self–acceleration

The pragmatic way (geodesic in the full metric) e.g. applied to radial fall

\[
\frac{d^2 \hat{z}}{dt^2} = f_{\hat{r}r}^t \left( \frac{d\hat{z}}{dt} \right)^3 + (2f_{\hat{r}t}^t - f_{\hat{r}rr}) \left( \frac{d\hat{z}}{dt} \right)^2 + (f_{\hat{t}t}^t - 2f_{\hat{r}tr}) \left( \frac{d\hat{z}}{dt} \right) - f_{\hat{r}tt}^t
\]

\[\implies \Delta \ddot{z} = \begin{cases} f_0 \Delta \dot{z} + f_1 \Delta z \end{cases} + f_2 \begin{cases} H_{1}^{ren}, H_{2}^{ren}, \partial H_{1}^{ren}, \partial H_{2}^{ren} \end{cases}
\]

Background geodesic deviation

Self–acceleration
Evolution approaches

Blue = particle position, Yellow = point where the SF is computed, Green = Blue + Yellow

1. Background geodesic

2. Energy balance

3. SF effect added after a stretch

4. 1st (any) order perturbation

5. Osculating orbit (geodesic at each point)

6. SF evolution (from previous point)

7. Self-consistent à la GW (all history considered)
Towards a self-consistent orbital evolution for EMRIs
First steps in Regge-Wheeler gauge for radial orbits

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In the Extreme Mass Ratio Inspirals (EMRIs) two-body problem, the determinant role of the wave equation is enshrined by the self-force, i.e., the self-consistent evolution. The latter, however, suffers from significant difficulties that cannot be evaded. The project wishes to sketch beyond the first applications, up to encompassing any non-adiabatic shift in non-rotating massive Black Holes (MBH) gravitational waveform. It is based on the AdS/CFT correspondence, which allows to transform the Dirac equation on the radial null foliation at the black hole horizon into a quantum field theory on AdS space. We use the Regge-Wheeler gauge formalism where the wavefunction and its derivatives. Such discontinuities may be derived from the singularity of the source term (odd/even) perturbations in RW gauge.

The perturbations may be written in terms of the gauge invariant, Rindler-AdS scalar wave function. Classically, the latter is a function of the perturbations shown in the form of the wave equation, where \( \psi_l^\nu_m \) serves as a wavefunction and its derivatives are used as guiding and reference throughout the integration.

Fourth order accuracy has been reached to compute the metric perturbations \( \tilde{h}_{\nu\rho} \) and \( \tilde{h}_{\mu\nu} \) with their first derivatives (classically implying the third order derivatives of \( \tilde{h}_{\nu\rho} \)).

\[ \tilde{h}_{\nu\rho} \approx \psi_{l,m}^\nu_m + \frac{1}{3} \psi_{l,m}^\nu_m \psi_{l,m}^\nu_m + \frac{1}{15} \psi_{l,m}^\nu_m \psi_{l,m}^\nu_m \psi_{l,m}^\nu_m \]

hence

\[ \tilde{h}_{\rho\nu} \approx \psi_{l,m}^\nu_m + \frac{1}{3} \psi_{l,m}^\nu_m \psi_{l,m}^\nu_m + \frac{1}{15} \psi_{l,m}^\nu_m \psi_{l,m}^\nu_m \psi_{l,m}^\nu_m \]

The leading-order of the metric perturbations on the trajectory shown as a peak in the value of the metric perturbations throughout the integration.

Gravitational waveforms for generic orbits
Indirect numerical method using the jump conditions

S. Aoudia, S. Cordier, P. Ritter, A. Spallicci

In Regge-Wheeler gauge, perturbations of a Schwarzschild-Dekker black hole of mass \( M \) induced by a point particle of mass \( m \) may be cast into the form of a wave equation, and hence of the wave function of the wave function. We numerically solve the above wave equation (source perturbations) and Regge-Wheeler (null particle) equations in time-domain and obtain the gravitational waveforms of arbitrary generic orbits \( \nu_l^\nu_m \). The wave equations are given by

\[ \psi_{l,m}^\nu_m = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}^{\nu_m} \partial_\mu \psi_{l,m}^\nu_m) - \frac{\kappa}{\sqrt{-g}} \psi_{l,m}^\nu_m \]

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where \( \Delta_r \rightarrow \infty \), \( \psi_r \rightarrow 0 \) and \( \sigma_r \rightarrow 0 \) are constant depending on the map the particle to an effective wavelength. The procedure is a classical finite difference scheme of order \( \kappa \).

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Fundamentals: charge e in flat spacetime

The charge-current density $j^\alpha$ is associated to $A^\alpha$, potential diverging on the world line, through

$$\Box A^\alpha = -4\pi j^\alpha \quad \partial_\alpha A^\alpha = 0 \quad \text{Lorenz gauge}$$

e orbits a centre, emits outgoing radiation, undergoes radiation reaction and spirals inward: $A^\alpha_{\text{ret}},$ retarded and singular, \textit{physically relevant solution} of $\Box A^\alpha$.

A radiation propagating inward and the charge spiraling outward: $A^\alpha_{\text{adv}},$ the advanced and singular, \textit{physically non-relevant solution} of $\Box A^\alpha$. Their sum

$$A^\alpha_S = \frac{1}{2} (A^\alpha_{\text{ret}} + A^\alpha_{\text{adv}})$$

- Outgoing and incoming radiation are equal: no loss nor gain of energy by the system.
- The potential $A^\alpha_S$ is locally isotropic around the particle and it does not exert a force on the charge (no radiation reaction).
- 'S' stands for 'symmetric', 'standing' and 'singular' (the three potentials diverge at the particle).

Removing $A^\alpha_S$ from $A^\alpha_{\text{ret}}$

$$A^\alpha_R = A^\alpha_{\text{ret}} - A^\alpha_S = \frac{1}{2} (A^\alpha_{\text{ret}} - A^\alpha_{\text{adv}}) \quad \Box A^\alpha_R = 0$$

- The potential $A^\alpha_R$ exerts a force on the particle.
- 'R' stands for 'regular' and 'radiative' (free radiation field).

For the e.m. field tensor $F^R_{\alpha\beta} = \partial_\alpha A^R_\beta - \partial_\beta A^R_\alpha$, the particle equations of motion is

$$m a_\mu = f^\text{ext}_\mu + e F^R_{\mu\nu} u^\nu = f^\text{ext}_\mu + \frac{2e^2}{3m} (\delta^\nu_{\mu} + u^\nu u_\mu) \frac{dr^\text{ext}_\nu}{d\tau}$$
Fundamentals: where $h^*_{\mu\nu}$ comes from? à la DeWh

\[ G_S^\alpha = \frac{1}{2} (G_{\text{ret}}^\alpha + G_{\text{adv}}^\alpha) \]
\[ G_R^\alpha = G_{\text{ret}}^\alpha - G_S^\alpha = \frac{1}{2} (G_{\text{ret}}^\alpha - G_{\text{adv}}^\alpha) \]

\[ G_S^\alpha = \frac{1}{2} (G_{\text{ret}}^\alpha + G_{\text{adv}}^\alpha - H) \]
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- In flat spacetime, the retarded potential at $x$ depends on the particle state of motion at the retarded point $z(u)$ on the world line; the advanced potential depends on the state of motion at the advanced point $z(v)$.
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In flat spacetime, the retarded potential at $x$ depends on the particle state of motion at the retarded point $z(u)$ on the world line; the advanced potential depends on the state of motion at the advanced point $z(v)$.

In curved spacetime, e.m. waves propagate at all speeds smaller than or equal to $c$ due to the interaction between the radiation and the spacetime curvature. The retarded potential at $x$ depends on the particle entire past history before $u$; the advanced potential depends on the particle entire future history after $v$.
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- In flat spacetime, the retarded potential at $x$ depends on the particle state of motion at the retarded point $z(u)$ on the world line; the advanced potential depends on the state of motion at the advanced point $z(v)$.
- In curved spacetime, e.m. waves propagate at all speeds smaller than or equal to $c$ due to the interaction between the radiation and the spacetime curvature. The retarded potential at $x$ depends on the particle entire past history before $u$; the advanced potential depends on the particle entire future history after $v$.
- The sum of half-retarded plus half-advanced is symmetric (solution to w.e.) but the difference (solution to the homogeneous w.e.) has an unacceptable dependence on the particle future history (non-causality).
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\[ G_S^\alpha = \frac{1}{2} (G_{\text{ret}}^\alpha + G_{\text{adv}}^\alpha - H) \]
\[ G_R^\alpha = G_{\text{ret}}^\alpha - G_S^\alpha = \frac{1}{2} (G_{\text{ret}}^\alpha - G_{\text{adv}}^\alpha + H) \]

- In flat spacetime, the retarded potential at $x$ depends on the particle state of motion at the retarded point $z(u)$ on the world line; the advanced potential depends on the state of motion at the advanced point $z(v)$.
- In curved spacetime, e.m. waves propagate at all speeds smaller than or equal to $c$ due to the interaction between the radiation and the spacetime curvature. The retarded potential at $x$ depends on the particle entire past history before $u$; the advanced potential depends on the particle entire future history after $v$.
- The sum of half-retarded plus half-advanced is symmetric (solution to w.e.) but the difference (solution to the homogeneous w.e.) has an unacceptable dependence on the particle future history (non-causality).
- The ad-hoc function $H$ determines that $G_S^\alpha(x)$ depends on the particle motion in the interval $u < \tau < v$.
- The ad-hoc function $H$ determines that $G_R^\alpha(x)$ depends on the particle motion in the interval $\tau < v$. The non-causal stretch $u < \tau < v$ disappears for $x \to z(\tau)$.
Fundamentals: where $h_{\mu\nu}^*$ comes from? à la MiSaTaQuWa

The FULL metric perturbation has 2 parts:

**DIRECT/INSTATANEOUS**

- It propagates away to infinity
- Support *only on* the past-null cone of the field point
- Diverges at the particle

**TAIL**

- It interacts with the spacetime curvature
- Support *inside* the past-null cone of the field point (particle history)
- Regular at the particle
- It gives the physical self-force