Statistical analysis of LISA Pathfinder noise

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Why we need a statistical tool for noise analysis?

Daily noise investigation

Analysis of fit residuals

Quantitative noise analysis

Statistical analysis of noise in spectral domain
Starting from the beginning

Welch's Overlapped Segment Averaging

Breaking up a time series into windowed overlapping segments.

Averaging the spectra for the different segments.

Allows to lower spectrum variance by a factor $N_{ave}^2$

Donald B. Percival and Andrew T. Walden,
SPECTRAL ANALYSIS FOR PHYSICAL APPLICATIONS,
Cambridge University Press.
The Statistic of the sample spectrum is a Gamma

Welch’s overlapped segment averaging (WOSA) spectrum \( P_{\text{WOSA}}(f_k) \)

\[
f_{\text{WOSA}}(z; h, \delta) = \frac{z^{(h-1)} e^{-\frac{z}{\delta}}}{\delta^h \Gamma(h)}
\]

\( \delta = \frac{S(f_k)}{N_{\text{ave}}} \)

\( S(f_k) = E[P_{\text{WOSA}}(f_k)] \)

\( h = N_{\text{ave}} \)

Normalized WOSA spectrum \( R_{\text{WOSA}}(f_k) = \frac{P_{\text{WOSA}}(f_k)}{S(f_k)} \)

Still a Gamma distribution with

\( \delta = 1/N_{\text{ave}} \)

\( h = N_{\text{ave}} \)
Overlapping segments are correlated

Averaging introduces a bias in the variance of the estimator

Normalized correction to the variance introduced by segment overlapping

50% overlap is a good tradeoff at each average value

Blackman-Harris window
Frequency bins are correlated

Correlation introduced by windowing

Blackman-Harris window

$10^5$ data points

50% overlap
Kolmogorov-Smirnov noise excess estimator

\[ d_K = \max |F_N(x) - F(x)| \]

Test cumulative function

Reference cumulative function

Kolmogorov-Smirnov noise excess estimator

\[ d_K = \max \left| F_N(x) - F(x) \right| \]

- Test cumulative function
- Reference cumulative function

If \( d_K > d_K(\alpha) \), the null hypothesis is rejected

\[ \alpha = P_{FA} \]

the null hypothesis is that the two CDFs are asymptotically equal

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The test works for uncorrelated data!!!
Kolmogorov-Smirnov noise excess estimator

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The test works for uncorrelated data!!!

overlap bias \( \rightarrow \) choice of the proper overlap %
window correlations \( \rightarrow \) calculate a correction factor by Monte Carlo on white noise

Test sensitivity - Data vs Model

Kolmogorov-Smirnov test

Likelihood Ratio test

\( \gamma \) is the minimum detectable noise energy excess
## Pros and Cons

<table>
<thead>
<tr>
<th>KS Test Pros</th>
<th>Likelihood Ratio Test Pros</th>
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<tbody>
<tr>
<td>• Independent from the data distribution</td>
<td>• Optimal, highest sensitivity</td>
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<tr>
<td>• Easy to use</td>
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<tr>
<td>• Compare data vs. data and data vs. model</td>
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<table>
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<tr>
<th>KS Test Cons</th>
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<tr>
<td>• Sub-optimal, less sensitive than the likelihood ratio</td>
<td>• Unpractical for data vs. data comparison</td>
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<td>• Once we depart from the simple case of constant excess over the frequency band, the distribution of data is unknown and the test is hardly implemented</td>
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Application to LTP
Application to LTP

Reference Spectrum
Application to LTP

Energy Excess = 1.041

The test does not reject the hypothesis that the two spectra are compatible.
Application to LTP

Energy Excess = 1.157

The test rejects the hypothesis that the two spectra are compatible.
Noise parameter estimation

Write a parametrized WOSA estimator for the normalized sample spectrum $R_{WOSA}(f_k; \theta)$.

The closer $\theta$ is to $\theta_{true}$, the better the distribution is described by a Gamma.

$$d_K(\theta) = \max |F_R(x; \theta) - P(N_s, xN_s)|$$

$F_R(x; \theta)$ is the ECDF for the current parameter estimate.

$P(N_s, xN_s)$ is the limiting CDF.

The minimization of $d_K(\theta)$ vs. $\theta$ provides the KS estimation for the parameter.

LTP Example - Grid search

True value

Non-stationary noise

3 days interferometer data @ 10 Hz
Non-stationary noise on capacitive actuators linearly increasing with the time
Non-stationary noise

Acceleration noise on the differential channel

Noise level increasing with the time
Non-stationary noise - KS test

Restrict to a frequency range of interest and run the test

Cut slices at given times and compare one another
Non-stationary noise - KS test

Rising above the thresholds

No clear variation
Thank you for your attention