Gravitational waves energy spectrum of hyperbolic encounters

Lorenzo De Vittori - ITP University of Zurich

In collaboration with Philippe Jetzer and Antoine Klein

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Hyperbolic encounters and GWs

- What do GWs from hyperbolic encounters look like?
- How much energy and power do they carry out of the system?
- Which frequencies do they have? Where are the peaks?
- Can they be detected and useful for the astrophysics of eLISA?
Already-known quantities - I

**Peters & Mathews:** the energy spectrum for keplerian orbits:

\[
P(n) = \frac{32 G^4 M_1^2 M_2^2 (M_1 + M_2)(1 - \varepsilon)^5}{5 c^5 \frac{r_p^5}{r_p^5}} \cdot g(n, \varepsilon) \quad (J_{\nu}(z))
\]

**Berry & Gair:** the energy spectrum for the parabolic limit \(\varepsilon \rightarrow 1:\)

\[
P(f) = \frac{4 \pi^2 G^3 M_1^2 M_2^2}{5 c^5 \frac{r_p^2}{r_p^2}} \cdot \ell(f) \quad (K_{\nu}(z))
\]

**Turner:** the energy spectrum for the limit \(\varepsilon \rightarrow \infty:\)

\[
P(\sigma) = \frac{G^{7/2} \frac{m_1^{1/2}}{m_1} \frac{m_2}{m_2}}{c^5 \frac{r_3.5}{r_p^3.5}} \frac{8}{15\pi} \sqrt{\varepsilon} \cdot T(\sigma) \quad (K_{\nu}(z))
\]
Already-known quantities - II

Capozziello et al.: total released energy from hyperbolic paths:

\[ E = \frac{32 \, G \, \mu^2 \, v_0^5}{b \, c^5} \cdot F(\varphi_0) \quad (\varepsilon) \]

Turner: total released energy from hyperbolic paths:

\[ E = \frac{8 \, G^{7/2} \, m_1^{1/2} \, m_2 \, m_2^2}{15 \, c^5 \, r_p^{3.5}} \cdot g(\varepsilon) \quad (\varepsilon) \]

Energies are computed integrating the power emitted per unit angle
Deriving the formula

Starting from the quadrupole formula \( P = \gamma \langle \dddot{D}_{ij}(t) \dddot{D}_{ij}(t) \rangle \), we make use of Parseval’s theorem - \( \int |f(t)|^2 dt = \int |\mathcal{F}(\omega)|^2 d\omega \):

\[
\Delta E = \int P(t) dt = \int \gamma \langle \dddot{D}_{ij}(t) \dddot{D}_{ij}(t) \rangle dt = \int \frac{\gamma \langle \dddot{D}_{ij}(\omega) \dddot{D}_{ij}(\omega) \rangle}{P(\omega)} d\omega
\]

Compute \( D_{ij}(t) \), the Fourier transform \( \dddot{D}_{ij}(\omega) \), and \( \langle \dddot{D}_{ij}(\omega) \dddot{D}_{ij}(\omega) \rangle \)

For the Fourier transform of \( D_{ij}(t) \) we used the procedure of Landau & Lifshitz for the energy spectrum analysis of a similar setting in electrodynamics.
The formula for the energy spectrum

\[ P(\omega) = -\frac{G a^4 m^2 \pi^2}{720 \, c^5} \omega^4 F_\varepsilon(\omega) \]

where the function \( F_\varepsilon(\omega) \) is:

\[
\left| [16 \varepsilon H_{i\nu}^{(1)'}(i\nu\varepsilon) + (\varepsilon^2 - 3) H_{i\nu}^{(1)'}(i\nu\varepsilon/2)] \right|^2 + \\
\left| [(3 - 2 \varepsilon^2) H_{i\nu}^{(1)'}(i\nu\varepsilon/2) - 8 \varepsilon H_{i\nu}^{(1)'}(i\nu\varepsilon)] \right|^2 + \\
\left| [8 \varepsilon H_{i\nu}^{(1)'}(i\nu\varepsilon) + \varepsilon^2 H_{i\nu}^{(1)'}(i\nu\varepsilon/2)] \right|^2 + \\
\left| \frac{9 (\varepsilon^2 - 1)}{\varepsilon^2} \left[ H_{i\nu}^{(1)}(i\nu\varepsilon/2) - 4 \varepsilon H_{i\nu}^{(1)}(i\nu\varepsilon) \right] \right|^2
\]

Hankel functions

\[ H_{\alpha}^{(1)}(z) = J_{\alpha}(z) + i Y_{\alpha}(z) \]
Numerical integration

We checked numerically that $\int P(\omega) d\omega = \Delta E$, i.e. that

$$\int_0^\infty \frac{G a^4 m^2 \pi^2}{720 c^5} \omega^4 F_\epsilon(\omega) \, d\omega = \frac{32 G \mu^2 v_0^5}{b c^5} \cdot F(\varphi_0)$$

our energy spectrum

energy emitted as found by Capozziello et al.

$$= \frac{8 G^{7/2} m_1^{1/2} m_2^2}{15 c^5 r_p^{7/2}} \cdot g(\epsilon)$$

energy emitted as found by Turner

Since we found an analytical identity between these two $\Delta E$.
The limit of large $\varepsilon$

Turner uses some approximations for $g(\varepsilon)$ and for the trajectories in the limit of $\varepsilon \gg 1$. Thus he can compute the power spectrum

$$P(\sigma) = \frac{G^{7/2} m_1^{1/2} m_2^2}{c^5 r_p^{3.5}} \frac{8}{15\pi} \sqrt{\varepsilon} \cdot \mathcal{T}(\sigma) \quad \text{where} \quad \sigma = \omega \tau$$

In that limit our results coincide.
The limit of $\varepsilon = 1$

According to Berry & Gair: $P(f) = \frac{4 \pi^2 G^3 M_1^2 M_2^2}{5 c^5 r_p^2} \cdot \ell(f)$.

$\ell(f)$ is in terms of modified Bessel functions of the second kind, i.e. $K_\nu(z)$, and in our result $F_\varepsilon(\omega)$ is instead in terms of Hankel functions, i.e. $H_\nu(z)$, also a combination of Bessel functions.

We compared the two results numerically and found that they do agree taking the parabolic limit of $\varepsilon \to 1$.
Conclusions

- We found an exact expression for the energy spectrum in the case of hyperbolic encounters.

- The formula is in agreement with the previous known results in the limits of $\varepsilon \to 1$ and $\varepsilon \to \infty$, as well as for $E = \int P$.

- Our results could be useful for a rough estimate of the parameters of the emitting systems.