Gravitational wave phasing for spinning binaries

A. Gopakumar, TIFR

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Plan of the talk

- An attempt to obtain temporally evolving GW polarization states $h_{\times,+}(t)$ for spinning binaries inspiralling along eccentric orbits
- How to do GW phasing for such binaries
- Why did we argue that an approach by Cornish & Key is not quite appropriate for the above purpose

In collaboration with Gerhard Schäfer
Motivation

- It is desirable to develop a prescription that can provide gravitational wave (GW) templates for spinning compact binaries inspiralling along eccentric orbits.

- A plausible application in the near future may be to model GWs from such binaries relevant for Pulsar Timing Array.

- GWs from systems like OJ287 (modelled to contain a spinning SMBH binary) should be interesting from the nano-Hz GW window.
GW phasing: I

- Aim is to obtain $h_{\times, +}(t)$ for spinning binaries inspiralling along eccentric orbits

- Consider $\times$ GW polarization state (Quadrupole in Amplitude)

  $$h_{\times} \bigg|_{Q} = \frac{4G\mu}{c^4 R^4} \left[ (p \cdot v)(q \cdot v) - \frac{Gm}{r} (p \cdot n)(q \cdot n) \right],$$

- The observer triad $[p, q, N]$ consists of the radial vector to the observer $N$, $q = N \times p$ & $p = N \times j_0$

  $j_0$: a unit vector along the total angular momentum at the initial epoch
  $r = r n$ & $v = dr/dt$

- Aim is to develop a prescription to obtain temporal evolution of these dot products via finding a way to obtain $r(t)$ & $v(t)$
GW phasing : II

- It is straightforward to write down \( \mathbf{r} \) in the \((p, q, N)\) frame:

\[
\mathbf{r} = r \left\{ \begin{bmatrix} \cos \Phi \cos \alpha - \sin \Phi \cos \iota \sin \alpha \end{bmatrix} \mathbf{p} + \left[ \begin{bmatrix} -\sin \iota S_\nu \\
+ \cos \iota \cos \alpha C_\nu \end{bmatrix} \sin \Phi + \cos \Phi \sin \alpha C_\nu \end{bmatrix} \mathbf{q} + \left[ \begin{bmatrix} \sin \iota C_\nu \\
+ \cos \iota \cos \alpha S_\nu \end{bmatrix} \sin \Phi + \cos \Phi \sin \alpha S_\nu \end{bmatrix} \mathbf{N} \right\},
\]

& a similar expression for \( \mathbf{v} \)

- Precession-al motion of \( \mathbf{k} \), a unit vector along \( \mathbf{L} \), will provide conservative \( \alpha(t) \) & \( \iota(t) \) as \( \mathbf{k} \) in the invariant frame, defined w.r.t \( \mathbf{j}_0 \):

\[
\mathbf{k} = \sin \iota \sin \alpha \mathbf{e}_x - \sin \iota \cos \alpha \mathbf{e}_y + \cos \iota \mathbf{e}_z
\]

In practice, one solves precessional equations for \( \mathbf{k}, s_1 \) and \( s_2 \) together.
GW phasing : II

- It is straightforward to write down \( r \) in the \((p, q, N)\) frame

\[
\begin{align*}
  r &= r \left\{ \left[ \cos \Phi \cos \alpha - \sin \Phi \cos \iota \sin \alpha \right] p + \left[ -\sin \iota S_\nu \right. \\
  &\quad \left. + \cos \iota \cos \alpha C_\nu \right] \sin \Phi + \cos \Phi \sin \alpha C_\nu \right] q + \left[ \left( \sin \iota C_\nu \\
  &\quad + \cos \iota \cos \alpha S_\nu \right) \sin \Phi + \cos \Phi \sin \alpha S_\nu \right] N \right\},
\end{align*}
\]

& a similar expression for \( v \)

- Precession-al motion of \( k \), a unit vector along \( L \), will provide conservative \( \alpha(t) \) & \( \iota(t) \) as \( k \) in the invariant frame, defined w.r.t \( j_0 \)

\[
  k = \sin \iota \sin \alpha e_x - \sin \iota \cos \alpha e_y + \cos \iota e_z
\]

In practice, one solves precessional equations for \( k, s_1 \) and \( s_2 \) together

- We need ways to obtain conservative evolutions for \( r(t) \) & \( \Phi(t) \)

After which one may impose effects of gravitational RR on the underlying conservative dynamics
Conservative radial dynamics: I

Take binaries containing two spinning compact objects \((m_1, m_2, S_1, S_2)\)

We want to obtain \(r(t)\) while considering dynamics that include effects of dominant order spin-orbit interactions

\[
H(r, p, S_1, S_2) = \frac{p^2}{2} - \frac{1}{r} + \frac{(L \cdot S_{\text{eff}})}{c^2 r^3},
\]

Scaled variables & effective spin

It is easy to obtain an expression for \(\dot{r} = p_r\)

\[
p_r^2 = 2E + \frac{2}{r} - \frac{L^2}{r^2} - \frac{2(L \cdot S_{\text{eff}})}{c^2 r^3}
\]

We can show that \(E, L, (L \cdot S_{\text{eff}})\) are conserved under the above Hamiltonian

Further, \(\dot{r}^2\) also is a polynomial in \(1/r\) with constant coefficients
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\]

- We can show that \(E, L, (L \cdot S_{\text{eff}})\) are conserved under the above Hamiltonian
- Further, \(\dot{r}^2\) also is a polynomial in \(1/r\) with constant coefficients
- It is possible to obtain Keplerian type parametric solution
Conservative radial dynamics: II

- Radial part of the orbital dynamics is given by

\[ r = a_r (1 - e_r \cos u), \]
\[ l \equiv n(t - t_0) = u - e_t \sin u, \]

where \( u \) and \( l \) are the eccentric and mean anomalies.

The orbital elements can be explicitly given in terms of \( E, L \) and \( L \cdot S_{\text{eff}} \).

- Our approach allows one to include (very easily) non-spinning PN corrections

\[ r = a_r (1 - e_r \cos u), \]
\[ l \equiv n(t - t_0) = u - e_t \sin u + \frac{g_{4t}}{c^4} (v - u) + \frac{f_{4t}}{c^4} \sin v, \]

These expressions provide 2PN-accurate orbital dynamics (radial part) that include dominant order spin-orbit interactions.
Conservative angular dynamics: I

- Angular part of our orbital dynamics is governed

\[
r \dot{\theta} = \mathbf{e}_\theta \cdot \dot{\mathbf{r}} = p_\theta + \frac{\mathbf{e}_\phi \cdot \mathbf{S}_{\text{eff}}}{c^2 r^2},
\]

\[
r \sin \theta \dot{\phi} = \mathbf{e}_\phi \cdot \dot{\mathbf{r}} = p_\phi - \frac{\mathbf{e}_\theta \cdot \mathbf{S}_{\text{eff}}}{c^2 r^2},
\]

where \( \theta \) is the angle between \( \mathbf{n} \) and \( \mathbf{j}_0 \) & \( \phi \) is the usual azimuthal angle [defined in the \( \mathbf{e}_x - \mathbf{e}_y \) plane, while \( \mathbf{j}_0 \) is along \( \mathbf{e}_z \)]

- The longitudinal & azimuthal motions are intertwined & it is rather impossible to tackle them analytically (without making further assumptions)
Conservative angular dynamics: I

- Angular part of our orbital dynamics is governed

\[ r \dot{\theta} = e_\theta \cdot \dot{r} = p_\theta + \frac{e_\phi \cdot S_{\text{eff}}}{c^2 r^2}, \]

\[ r \sin \theta \dot{\phi} = e_\phi \cdot \dot{r} = p_\phi - \frac{e_\theta \cdot S_{\text{eff}}}{c^2 r^2}, \]

where \( \theta \) is the angle between \( n \) and \( j_0 \) & \( \phi \) is the usual azimuthal angle [defined in the \( e_x - e_y \) plane, while \( j_0 \) is along \( e_z \)]

- The longitudinal & azimuthal motions are intertwined & it is rather impossible to tackle them analytically (without making further assumptions)
- In other words, it is difficult to find \( r(t) \) semi-analytically

\[ r(t) = r(t) n(t), \text{ where} \]

\[ n(t) = \sin \theta \cos \phi e_x + \sin \theta \sin \phi e_y + \cos \theta e_z, \]
However, $h_{x,+}$ depend on the usual 3 Eulerian angles $(\Phi, \alpha, i)$ such that $r = r \mathbf{n}$

$$
\mathbf{n} = \left\{ \begin{array}{c}
(c \cos \Phi \cos \alpha - s \sin \Phi \cos i \sin \alpha) \mathbf{e}_x \\
+ (c \cos \Phi \sin \alpha + s \sin \Phi \cos i \cos \alpha) \mathbf{e}_y + s \Phi \sin i \mathbf{e}_z
\end{array} \right\},
$$

and hence we only need to find a way to obtain $\Phi(t)$.
However, $h_{\times,+}$ depend on the usual 3 Eulerian angles $(\Phi, \alpha, i)$ such that $r = r n$

$$n = \left\{ \begin{array}{l}
(\cos \Phi \cos \alpha - \sin \Phi \cos i \sin \alpha) \ e_x \\
+ (\cos \Phi \sin \alpha + \sin \Phi \cos i \cos \alpha) \ e_y + \sin \Phi \sin i \ e_z \end{array} \right\},$$

and hence we only need to find a way to obtain $\Phi(t)$.

Recall that $r(t)$ is under control; while $i(t)$ & $\alpha(t)$ via $\dot{k}, \dot{s}_1, \dot{s}_2$.
However, $h_{x,+}$ depend on the usual 3 Eulerian angles $(\Phi, \alpha, i)$ such that $r = r \, n$

$$n = \left\{ (\cos \Phi \cos \alpha - \sin \Phi \cos \iota \sin \alpha) \, e_x \right.$$\n\n$$+ (\cos \Phi \sin \alpha + \sin \Phi \cos \iota \cos \alpha) \, e_y + \sin \Phi \sin \iota \, e_z \right\} ,$$

and hence we only need to find a way to obtain $\Phi(t)$

Recall that $r(t)$ is under control; while $i(t)$ & $\alpha(t)$ via $\dot{k}, \dot{s}_1, \dot{s}_2$

The idea is to write $v = \dot{r}$ in a co-moving triad $(n, \xi = k \times n, k)$:

$$v = \dot{r} \, n + r \left( \frac{d\Phi}{dt} + \frac{d\alpha}{dt} \cos \iota \right) \, \xi + r \left( \frac{d\iota}{dt} \sin \Phi - \sin \iota \cos \Phi \frac{d\alpha}{dt} \right) \, k .$$
The expressions for $\nu^2 = \mathbf{v} \cdot \mathbf{v}$ arising from the above Equation & from our Hamiltonian allows us to obtain

\[ \dot{\Phi} = \frac{L}{r^2} + \frac{(\mathbf{k} \cdot \mathbf{S}_{\text{eff}})}{c^2 r^3} - \cos i \dot{\alpha}, \]

where $\mathbf{v}$ is the velocity vector, $L$ is the angular momentum, $\mathbf{k}$ is the angular momentum vector, $\mathbf{S}_{\text{eff}}$ is the effective angular momentum, $c$ is the speed of light, and $i$ is the inclination angle.
The expressions for $v^2 = \mathbf{v} \cdot \mathbf{v}$ arising from the above Equation & from our Hamiltonian allows us to obtain

$$\dot{\Phi} = \frac{L}{r^2} + \frac{(\mathbf{k} \cdot \mathbf{S}_{\text{eff}})}{c^2 r^3} - \cos i \dot{\alpha},$$

Recall that $\dot{\alpha}$ does not admit Keplerian type parametric solution. This $\rightarrow$ $\dot{\Phi}$ also will not have a semi-analytic solution ( & hence requires numerical treatment).

Our approach allows one to include non-spinning corrections to $\dot{\Phi}$ in an easy manner. This allows us to obtain fully 2PN-accurate evolution for $r(t)$ and $\Phi(t)$.
Including Reactive dynamics:

- Gravitational RR is included in an adiabatic manner. This is because RR time scale is longer than the time scales associated with the orbital and precessional motions.

- Therefore, the conserved quantities $E$ and $L$ appearing in the PN-accurate equations that govern $r(t)$, $\Phi(t)$ and $\alpha(t)$ are allowed to vary with time.

- It is natural to use far-zone energy & angular momentum fluxes to provide variations in $E$ and $L$.

- Further, it is also convenient to use $e_t$ and $\omega = \dot{\Phi} + \cos \iota \dot{\alpha}$ while writing down expressions for the far-zone fluxes.
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- Further, it is also convenient to use $e_t$ and $\omega = \dot{\Phi} + \cos\iota \dot{\alpha}$ while writing down expressions for the far-zone fluxes.

- This is our prescription to obtain $h_{x, \pm}(t)$ from spinning compact binaries in inspiralling eccentric orbits.
Including Reactive dynamics :II

- At present, we are working on a code that implements the above prescription to create easy-to-use $h_{\times,+}(t)$

- Developing a way to include spin-spin effects

- It will be desirable to probe data analysis implications
At present, we are working on a code that implements the above prescription to create easy-to-use $h_{\times,+}(t)$.

Developing a way to include spin-spin effects.

It will be desirable to probe data analysis implications.

This is because we initiated the project as Cornish & Key (2010) argued that angular part of the above dynamics is integrable.
Cornish & Key (CK) argued that *Angular Part of the above Orbital Dynamics* admits a Keplerian type parametric solution in a non-inertial frame that follows the precession of the orbital plane, defined by $\mathbf{L}_N = \mu \mathbf{r} \times \mathbf{v}$.

Recall that angular part of the orbital dynamics:

$$r \dot{\theta} = \mathbf{e}_\theta \cdot \dot{\mathbf{r}} = p_\theta + \frac{\mathbf{e}_\phi \cdot \mathbf{S}_{\text{eff}}}{c^2 r^2},$$

$$r \sin \theta \dot{\phi} = \mathbf{e}_\phi \cdot \dot{\mathbf{r}} = p_\phi - \frac{\mathbf{e}_\theta \cdot \mathbf{S}_{\text{eff}}}{c^2 r^2},$$

Their precessing frame is given by

$$\left. \frac{d\mathbf{I}_N}{dt} \right|_{\text{pre}} = 0 = \frac{d\mathbf{I}_N}{dt} - \frac{1}{c^2 r^3} (\mathbf{S}_{\text{eff}} \times \mathbf{I}_N),$$

where $\mathbf{I}_N = \mathbf{L}_N / |\mathbf{L}_N|$ & they introduced $\mathbf{v}_{\text{pre}}$. 
Orbital velocity in the precessional frame is

\[ \mathbf{v}_{\text{pre}} = \mathbf{v} - \mathbf{\omega} \times \mathbf{r}, \]

where \( \mathbf{\omega} = \mathbf{S}_{\text{eff}} / (c^2 r^3) \) implying that the radial motion is unchanged.

CK argued that \( \mathbf{L}_N \cdot \mathbf{v}_{\text{pre}} = 0 \)

Orbital plane remains fixed in the precessing frame ..like in the Newtonian dynamics.

However, we showed that \( \mathbf{L}_N \cdot \mathbf{v}_{\text{pre}} \neq 0 \) at the the considered PN order

\[ \mathbf{L}_N \cdot \mathbf{v}_{\text{pre}} = -\mu \left\{ (\mathbf{r} \times \mathbf{v}) \cdot (\mathbf{\omega} \times \mathbf{r}) \right\} \neq 0 \]
Orbital velocity in the precessional frame is

$$v_{\text{pre}} = v - \omega \times r,$$

where $$\omega = S_{\text{eff}}/(c^2 r^3)$$ implying that the radial motion is unchanged.

CK argued that $$L_N \cdot v_{\text{pre}} = 0$$
Orbital plane remains fixed in the precessing frame ..like in the Newtonian dynamics.

However, we showed that $$L_N \cdot v_{\text{pre}} \neq 0$$ at the the considered PN order

$$L_N \cdot v_{\text{pre}} = -\mu \left\{ \left( r \times v \right) \cdot \left( \omega \times r \right) \right\} \neq 0$$

Further, invoking $$L = r \times p$$ instead of $$L_N$$ to describe the orbital plane will NOT help.
GW phasing CK’s approach:

- Orbital velocity in the precessional frame is
  \[ \mathbf{v}_{\text{pre}} = \mathbf{v} - \mathbf{\omega} \times \mathbf{r}, \]
  where \( \mathbf{\omega} = \mathbf{S}_{\text{eff}}/(c^2 r^3) \) implying that the radial motion is unchanged.

- CK argued that \( \mathbf{L}_N \cdot \mathbf{v}_{\text{pre}} = 0 \)
  Orbital plane remains fixed in the precessing frame ..like in the Newtonian dynamics.

- However, we showed that \( \mathbf{L}_N \cdot \mathbf{v}_{\text{pre}} \neq 0 \) at the the considered PN order
  \[ \mathbf{L}_N \cdot \mathbf{v}_{\text{pre}} = -\mu \left\{ \left( \mathbf{r} \times \mathbf{v} \right) \cdot \left( \mathbf{\omega} \times \mathbf{r} \right) \right\} \neq 0 \]

- Further, invoking \( \mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \) instead of \( \mathbf{L}_N \) to describe the orbital plane will NOT help.

- This simply means that \( p_{\phi}^2 + p_{\theta}^2 = L^2/r^2 \)
How does one prove that angular part does NOT admit Keplerian type solution

According to CK, angular part admits Keplerian-type solution

\[ v_{\text{pre}} = \frac{d r'}{dt} \big|_{\text{pre}}, \text{ where } r'(t) = r(t) (\cos \varphi(t) p'(t) + \sin \varphi(t) l'(t)) \]

\[ p' = k \times N'/|k \times N'| & \quad l' = k \times p' \]

while \( k \) and \( N' \) are unit vectors along \( L \) and the line of sight vector to the source

Let us compute from \( r' \), \( v_{\text{pre}}^2 \) and see we get the expected

\[ v_{\text{pre}}^2 = \dot{r}^2 + r^2 \dot{\varphi}^2 \]
GW phasing CK’s approach:III

How does one prove that angular part does NOT admit Keplerian type solution

According to CK, angular part admits Keplerian-type solution
\[ v_{\text{pre}} = \frac{d\mathbf{r}'}{dt}|_{\text{pre}}, \text{ where } \mathbf{r}'(t) = r(t) \left( \cos \varphi(t) \mathbf{p}'(t) + \sin \varphi(t) \mathbf{l}'(t) \right) \]
\[ \mathbf{p}' = \mathbf{k} \times \mathbf{N}' / |\mathbf{k} \times \mathbf{N}'| \text{ & } \mathbf{l}' = \mathbf{k} \times \mathbf{p}' \]
while \( \mathbf{k} \) and \( \mathbf{N}' \) are unit vectors along \( \mathbf{L} \) and the line of sight vector to the source

Let us compute from \( \mathbf{r}' \), \( v^2_{\text{pre}} \) and see we get the expected
\[ v^2_{\text{prec}} = \dot{r}^2 + r^2 \dot{\varphi}^2 \]

\[ v^2_{\text{prec}} = \dot{r}^2 + r^2 \dot{\varphi}^2 + r \dot{r} \left[ (\mathbf{l}' \cdot \mathbf{n}') \sin \varphi + (\mathbf{p}' \cdot \mathbf{n}') \cos \varphi \right] + r^2 \dot{\varphi} (\mathbf{l}' \cdot \mathbf{p}') \]

It is trivial to show that \( \mathbf{p}' \cdot \mathbf{n}' \neq 0 \),
Similar arguments hold for \( \mathbf{l}' \cdot \mathbf{n}' \) & \( \mathbf{l}' \cdot \mathbf{p}' \).
Therefore, \( r' = r (\cos \varphi \mathbf{p}' + \sin \varphi \mathbf{l}') \) will not give a \( \mathbf{v}_{\text{pre}} \) such that
\[
\mathbf{v}_{\text{pre}}^2 = \dot{r}^2 + r^2 \dot{\varphi}^2
\]

This is true even while employing \( \mathbf{L} \) to define the orbit.

There are other issues associated with CK’s approach.

This is why we were forced to present our approach.
Summary

- We discussed how to develop a way to create easy-to-use GW search templates for spinning compact binaries inspiralling along eccentric orbits.

- It will be nice to probe its data analysis implications.
THANK YOU!