Quantum mechanics aspects and subtleties of neutrino oscillations

Evgeny Akhmedov

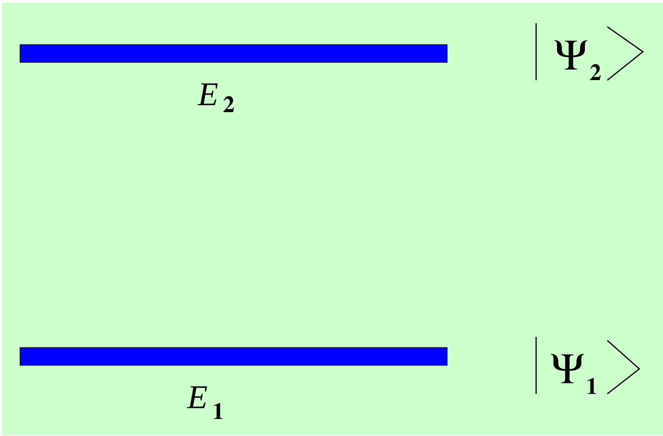
Max-Planck Institute für Kernphysik, Heidelberg

MAX-PLANCK-INSTITUT
FÜR KERNPHYSIK

Neutrino oscillations:

- A beautiful QM interference phenomenon
- Reveals many aspects of QM
- Owes its very existence to QM uncertainty relations
- Can actually be used to study QM!

Oscillations: a well known QM phenomenon



$$\Psi_1(t) = e^{-i\,E_1\,t}\,\Psi_1(0)$$

$$\Psi_2(t) = e^{-i\,E_2\,t}\,\Psi_2(0)$$

$$\Psi(0) = a\,\Psi_1(0) + b\,\Psi_2(0) \quad (|a|^2 + |b|^2 = 1)\;; \qquad \Rightarrow$$

$$\Psi(t) = a\,e^{-i\,E_1\,t}\,\Psi_1(0) + b\,e^{-i\,E_2\,t}\,\Psi_2(0)$$

Probability to remain in the same state $|\Psi(0)\rangle$ after time t:

◇ $$P_{\rm surv} = |\langle\Psi(0)|\Psi(t)\rangle|^2 = \Big||a|^2\,e^{-i\,E_1\,t} + |b|^2\,e^{-i\,E_2\,t}\Big|^2$$

$$= 1 - 4|a|^2|b|^2\sin^2[(E_2 - E_1)\,t/2]$$

Neutrino oscillations appear to be a simple QM phenomenon

But: A closer look at them reveals a number of subtle and even paradoxical issues

The idea of ν oscillations: put forward by B. Pontecorvo over 60 years ago

20 years have passed after the discovery of ν oscillations

Neutrino oscillation theory actively developed since the 1960s

A number of fundamental issues have long been (and still are) actively debated!

Debating the basics of neutrino oscillations ...

Shtanov hep-ph/9706378, Field hep-ph/0110064, hep-ph/0211199, arXiv:hep-ph/0401051, Lipkin arXiv:0801.1465, arXiv:0905.1216, arXiv:0910.5049, Ivanov & Kienle arXiv:0909.1287, Merle arXiv:0907.3554, Peshkin arXiv:0804.4891, Faber arXiv:0801.3262, Gal arXiv:0809.1213, Giunti arXiv:0805.0431, Flambaum arXiv:0908.2039, Kienert, Kopp, Lindner & Merle arXiv:0808.2389, Walker Nature 453 (2008) 864, Giunti arXiv:0807.3818, Kleinert & Kienle ("Neutrino-pulsating vacuum") arXiv:0803.2938, Lambiase, Papini & Scarpeta arXiv:0811.2302, Burkhardt, Lowe, Stephenson, Goldman & McKellar, arXiv:0804.1099 Bilenky, v. Feilitzsch & Potzel arXiv:0804.3409, arXiv:0803.0527, J. Phys. G36 (2009) 078002, EA, Kopp & Lindner arXiv:0802.2513, arXiv:0803.1424, Cohen, Glashow & Ligety arXiv:0810.4602, Visinelli & Gondolo arXiv:0810.4132, EA & Smirnov, arXiv:0905.1903, Keister & Polizou arXiv:0908.1404, Nishi & Guzzo arXiv:0803.1422, Lychkovskiy arXiv:0901.1198, Adhikari & Pal arXiv:0912.5266, Giunti arXiv:1001.0760, Ahluwalia & Schritt arXiv:0911.2965, Schmidt-Parzefall arXiv:0912.3620, EA & Kopp, arXiv:1001.4815, Robertson arXiv:1004.1847, Kayser, Kopp, Roberston & Vogel arXiv:1006.2372, Wu, Hutasoit, Boyanovsky & Holman arXiv:1002.2649, arXiv:1005.3260, Boyanovsky arXiv:1106.6248, Volobuev arXiv:1703.08070, Egorov & Volobuev arXiv:1709.09915

and many, many others.

Debated issues ("damned questions")

- Calculating the oscillation phase: Same E or same p ?
- Can we use plane waves or stationary states?
- Evolution in space or in time?
- Do the oscillations contradict energy-momentum conservation?
- Under what conditions can oscillations be observed?
  (coherence issues)
- What is the role of QM uncertainties?
- Is wave packet approach really necessary?
- What determines the size of the neutrino wave packet?

"Damned questions" – contd.

- When are the oscillations described by a universal probability?
- Is the standard oscillation formula correct? What is domain of its applicability?
- How to get correctly normalized oscillation probability?
- Are the oscillation probabilities Lorentz invariant? Can we see that?
- Why do we say that charged leptons are produced as mass eigenstates and neutrinos as flavour states and not the other way around?
- Do charged leptons oscillate?
- Do neutrinos produced in $\pi \to l\nu_l$ decays oscillate when the charged lepton is not detected?

Master formula for ν oscillations

The standard formula for the oscillation probability of relativistic or quasi-degenerate in mass neutrinos in vacuum:

$$\diamondsuit \qquad \boxed{P(\nu_\alpha \to \nu_\beta; L) = \left| \sum_i U_{\beta i}\, e^{-i\frac{\Delta m_{ij}^2}{2p} L}\, U_{\alpha i}^* \right|^2}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} P(\nu_\alpha \to \nu_\beta; L) = \delta_{\alpha\beta} - 4 \sum_{i<j} \mathrm{Re}[U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}] \sin^2\left(\frac{\Delta m_{ij}^2}{4p}\right) \\ + 2 \sum_{i<j} \mathrm{Im}[U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}] \sin\left(\frac{\Delta m_{ij}^2}{2p}\right) \end{aligned}}$$

How is it usually derived?

Assume at time $t = 0$ and coordinate $x = 0$ a flavour eigenstate $|\nu_\alpha\rangle$ is produced:

$$|\nu(0,0)\rangle \;=\; |\nu_\alpha^{\rm fl}\rangle \;=\; \sum_i U_{\alpha i}^*\,|\nu_i^{\rm mass}\rangle$$

After time t at the position x, for <u>plane-wave particles</u>:

$$|\nu(t,\vec{x})\rangle \;=\; \sum_i U_{\alpha i}^*\,e^{-ip_i x}|\nu_i^{\rm mass}\rangle$$

Mass eigenstates pick up the phase factors $e^{-i\phi_i}$ with

$$\phi_i \;\equiv\; p_i\,x \;=\; Et \;-\; \vec{p}\vec{x}$$

$$P(\nu_\alpha \to \nu_\beta) \;=\; \left|\langle\nu_\beta^{\rm fl}|\nu(t,x)\rangle\right|^2$$

How is it usually derived?

Phase differences between different mass eigenstates:

$$\Delta\phi \;=\; \Delta E\cdot t \;-\; \Delta \mathrm{p}\cdot \mathrm{x}$$

Shortcuts to the standard formula

1. Assume the emitted neutrino state has a well defined momentum (same momentum prescription) $\quad\Rightarrow\quad \Delta \mathrm{p} \;=\; 0.$

   For ultra-relativistic neutrinos $\;E_i \;=\; \sqrt{\mathrm{p}^2 + m_i^2} \simeq p + \frac{m_i^2}{2\mathrm{p}} \quad\Rightarrow$

$$\Delta E \;\simeq\; \frac{m_2^2 - m_1^2}{2E} \;\equiv\; \frac{\Delta m^2}{2E}\,; \qquad t \approx x \qquad (\hbar = c = 1)$$

$\Rightarrow$ The standard formula is obtained

How is it usually derived?

2. Assume the emitted neutrino state has a well defined energy (same energy prescription) $\Rightarrow$ $\Delta E = 0$.

$$\Delta\phi = \Delta E \cdot t - \Delta \mathrm{p} \cdot \mathrm{x} \quad \Rightarrow \quad - \Delta \mathrm{p} \cdot \mathrm{x}$$

For ultra-relativistic neutrinos $\mathrm{p}_i = \sqrt{E^2 - m_i^2} \simeq E - \frac{m_i^2}{2\mathrm{p}} \quad \Rightarrow$

$$-\Delta \mathrm{p} \equiv \mathrm{p}_1 - \mathrm{p}_2 \approx \frac{\Delta m^2}{2E}\,;$$

$\Rightarrow$ The standard formula is obtained

Stand. phase $\Rightarrow$ $\boxed{(l_{\mathrm{osc}})_{ik} = \frac{4\pi E}{\Delta m_{ik}^2}}$

Same E and same p approaches

Same E and same p approaches

Very simple and transparent

Same E and same p approaches

Very simple and transparent

Allow one to quickly arrive at the desired result

Same E and same p approaches

Very simple and transparent

Allow one to quickly arrive at the desired result

Trouble: they are both inconsistent

Kinematic constraints

Same momentum and same energy assumptions: contradict kinematics!

Easy to see for processes with 2-body final states:

- Electron capture: R. Winter, Lett. Nuovo Cim. 30 (1981) 101
- Pion decay: C. Giunti & C.W. Kim, Found. Phys. Lett. 14 (2001) 213

Kinematic constraints

Pion decay at rest $(\pi^+ \to \mu^+ + \nu_\mu, \quad \pi^- \to \mu^- + \bar{\nu}_\mu)$:
For decay with emission of a massive neutrino of mass m_i:

$$E_i^2 = \frac{m_\pi^2}{4}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 + \frac{m_i^2}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_i^4}{4m_\pi^2}$$

$$p_i^2 = \frac{m_\pi^2}{4}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 - \frac{m_i^2}{2}\left(1 + \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_i^4}{4m_\pi^2}$$

For massless neutrinos: $E_i = p_i = E \equiv \frac{m_\pi}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \simeq 30\ \mathrm{MeV}$

To first order in m_i^2:

$$E_i \simeq E + \xi\frac{m_i^2}{2E}\,, \qquad p_i \simeq E - (1-\xi)\frac{m_i^2}{2E}\,, \qquad \xi = \frac{1}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \approx 0.2$$

Kinematic constraints

Same momentum or same energy would require
$\xi = 1$ or $\xi = 0$ – not the case!

Also: would violate Lorentz invariance of the oscillation probability

How can wrong assumptions lead to the correct oscillation formula ?

$\Rightarrow$ Solution: Wave packet approach

Problems with the plane-wave approach

- Same momentum $\Rightarrow$ momentum is well defined (plane waves). Oscillation probabilities depend only on time. Leads to a paradoxical result – no need for a far detector !
  "Time-to-space conversion" (??) $x \approx t$ – assumes neutrinos to be point-like particles (notion opposite to plane waves).

- Same energy – oscillation probabilities depend only on coordinate. Does not explain how neutrinos are produced and detected at certain times. Corresponds to a stationary situation.

E and p conservation and ν production

Neutrino oscillations and energy-momentum conservation

– an intricate relationship

E and p conservation and ν production

Neutrino oscillations and energy-momentum conservation

– an intricate relationship

Calculation of rates of processes in quant. theory (gen. Fermi's Golden rule):

$$\Gamma = \prod_i \frac{1}{(2E_i)} \int \prod_f \frac{d^3 p_f}{(2\pi)^3\, 2E_f} |M_{fi}|^2 (2\pi)^4 \delta^4\Big(\sum_f p_f - \sum_i p_i\Big)$$

The factor $\delta^4\Big(\sum_f p_f - \sum_i p_i\Big)$ ensures energy-momentum conservation.

E and p conservation and ν production

Neutrino oscillations and energy-momentum conservation

– an intricate relationship

Calculation of rates of processes in quant. theory (gen. Fermi's Golden rule):

$$\Gamma = \prod_i \frac{1}{(2E_i)} \int \prod_f \frac{d^3 p_f}{(2\pi)^3\, 2E_f} \left|M_{fi}\right|^2 (2\pi)^4 \delta^4\left(\sum_f p_f - \sum_i p_i\right)$$

The factor $\delta^4\left(\sum_f p_f - \sum_i p_i\right)$ ensures energy-momentum conservation.

Used to calculate neutrino production rates and detection cross sections.

E and p conservation and ν production

Neutrino oscillations and energy-momentum conservation
– an intricate relationship

Calculation of rates of processes in quant. theory (gen. Fermi's Golden rule):

$$\Gamma = \prod_i \frac{1}{(2E_i)} \int \prod_f \frac{d^3 p_f}{(2\pi)^3\, 2E_f} |M_{fi}|^2 (2\pi)^4 \delta^4\Big(\sum_f p_f - \sum_i p_i\Big)$$

The factor $\delta^4\Big(\sum_f p_f - \sum_i p_i\Big)$ ensures energy-momentum conservation.

Used to calculate neutrino production rates and detection cross sections.

If applied to neutrino production, implies that the neutrino 4-momentum $p = (E, \vec{p})$ can be determined from the 4-momenta of all other particles participating in the production process.

A dichotomy

But: Due to the on-shell relation

$$E^2 = \vec{p}^{\,2} + m^2 ,$$

if the neutrino energy and momentum are exactly known, so is its mass!

$\Rightarrow$ The emitted neutrino is a mass eigenstate rather than a flavor eigenstate (= coherent superposition of different mass eigenstates).

$\Rightarrow$ Neutrino oscillations cannot occur! (Mass eigenstates do not oscillate in vacuum).

A dichotomy:

On the one hand, energy-momentum conservation is an exact law of nature.

On the other hand, exact energy and momentum conservation at neutrino production or detection would apparently make the oscillations impossible.

$\Rightarrow$ Significant confusion in the literature

QM uncertainties of energies and momenta

In the analyses of elementary particle processes it usually is assumed that energies and momenta of all particles have sharp values which obey conservation laws.

QM uncertainties of energies and momenta

In the analyses of elementary particle processes it usually is assumed that energies and momenta of all particles have sharp values which obey conservation laws.

But: for this picture to be exact, the processes (and the involved particles) should be completely delocalized in space and in time !

QM uncertainties of energies and momenta

In the analyses of elementary particle processes it usually is assumed that energies and momenta of all particles have sharp values which obey conservation laws.

But: for this picture to be exact, the processes (and the involved particles) should be completely delocalized in space and in time !

In reality the processes occur in finite spatial regions and during finite time intervals $\Rightarrow$

Energies and momenta of all participating particles have intrinsic QM uncertainties. Their states are not eigenstates of energy and momentum.

QM uncertainties of energies and momenta

In the analyses of elementary particle processes it usually is assumed that energies and momenta of all particles have sharp values which obey conservation laws.

But: for this picture to be exact, the processes (and the involved particles) should be completely delocalized in space and in time !

In reality the processes occur in finite spatial regions and during finite time intervals $\Rightarrow$

Energies and momenta of all participating particles have intrinsic QM uncertainties. Their states are not eigenstates of energy and momentum.

$\Rightarrow$ Does not mean that energy and momentum are not conserved ! Their conservation laws are fulfilled for the individual momentum components of the transition amplitudes.

Oscillations and QM uncertainty relations

Neutrino oscillations – a QM interference phenomenon, owe their existence to QM uncertainty relations

Neutrino energy and momentum are characterized by uncertainties σ_E and σ_p related to the spatial localization and time scale of the production and detection processes. These uncertainties

- allow the emitted/absorbed neutrino state to be a coherent superposition of different mass eigenstates
- determine the size of the neutrino wave packets $\Rightarrow$ govern decoherence due to wave packet separation

σ_E – the effective energy uncertainty, dominated by the smaller one between the energy uncertainties at production and detection. Similarly for σ_p.

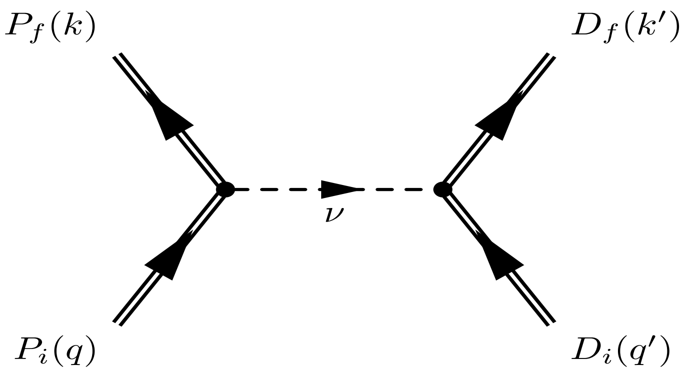
- Consistent approaches:

♢ Consistent approaches:

- QM wave packet approach – neutrinos described by wave packets rather than by plane waves

◇ Consistent approaches:

- QM wave packet approach – neutrinos described by wave packets rather than by plane waves
- QFT approach: neutrino production and detection explicitly taken into account. Neutrinos are intermediate particles described by propagators



Wave packet approach

S. Nussinov, Phys. Lett. 63B (1976) 201

1st discussion of neutrino oscillations in WP approach; decoherence by WP separation; $l_{\rm coh} \approx \sigma_x (v/\Delta v)$; estimates of σ_x and coher. length for ν production in decays of isolated nuclei and $\pi^\pm$ ($\sigma_x \approx c/\Gamma_{source}$) and also for solar neutrinos ($\rho \sim 100~{\rm g/cm^3}$, $T \sim 1~{\rm keV}$, $\sigma_x \approx c\tau$, τ – time of un-interrupted neutrino emission = time between collisions significantly changing the phase of the emitter; $\tau \sim 3\times 10^{-17}~{\rm s}$ for ^{7}Be neutrinos $\Rightarrow \quad l_{\rm coh} \sim 10~{\rm km}$.

B. Kayser, Phys. Rev. D(1981) 110

Production and detection coherence conditions: σ_E and σ_p must be sufficiently large to prevent accurate determination of neutrino mass; connection with space-time localization of the ν production and detection processes through QM uncertainty relations. A simplified analytic description of ν oscillations in the WP picture given.

Kobzarev, Martemyanov, Okun & Shchepkin, Sov. J. Nucl. Phys. 35 (1982) 708

Neutrino production and detection processes included in a simplified model.

Wave packet approach

Giunti, Kim & Lee, Phys. Rev. D44 (1991) 3635

First complete analytic derivation of $P_{\rm osc}$ for Gaussian WPs. Explicitly demonstrated how the oscillations get suppressed when coherence conditions are violated.

Giunti, Kim, Lee & Lee, Phys. Rev. D48 (1993) 4310

Neutrino production and detection processes included, source and target particles are localized – described by Gaussian WPs. QFT approach.

Rich, Phys. Rev. D48 (1993) 4318

Neutrino prod. and det. processes included, source and target particles are localized – described by Gaussian WPs (external WPs). QM approach.

Kiers & Weiss, Phys. Rev. D57 (1998) 3091

Neutrino production and detection included in a simple model with localized source and detector. Possible restoration of coherence at detection after decoherence by WP separation.

Wave packet approach

Farzan & Smirnov, Nucl. Phys. B805 (2008) 356 (arXiv:0803.0495)
Propagation decoherence in momentum space as effect of accumulation of fluctuations of $\phi_{\rm osc}$ due to momentum spread within WP with distance; unimportance of spatial spreading of ν WPs; Lorentz invariance of $\sigma_x E_\nu$.

EA & Smirnov, Phys. Atom. Nucl. 72 (2009) 1363 (arXiv:0905.1903)
Cleared up same E / same p confusion; shape-independent WP approach; Lorentz invariance of osc. probability in both coherent and decoherent cases demonstrated.

EA & J. Kopp, JHEP 1004 (2010) 008 (arXiv:1001.4815)
QM and QFT approaches compared; the issue of normalization of $P_{\rm osc}$ clarified; conditions for existence of the universal (production and detection indep.) oscillation probability found.

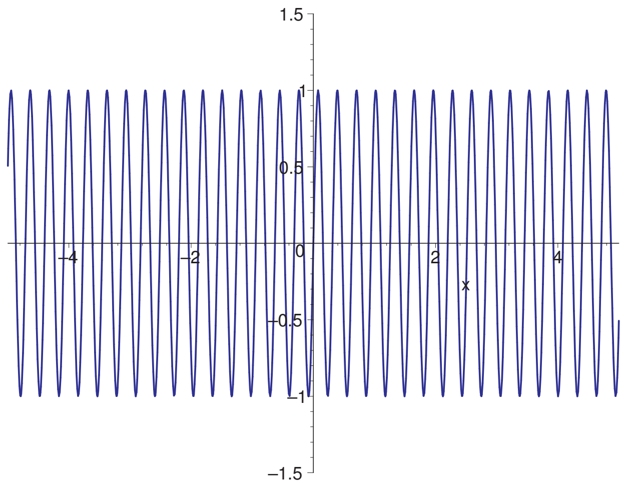
EA, JHEP 1707 (2017) 070 (arXiv:1703.08169)
The issue of whether non-relativistic neutrinos oscillate clarified; Lorentz invariance discussed.

QM wave packet approach

In QM propagating particles are described by wave packets!

– Finite extensions in space and time.

Plane waves: the wave function at time $t = 0$ $\quad \Psi_{\vec{p}_0}(\vec{x}) \;=\; e^{i\vec{p}_0\vec{x}}$



Wave packets

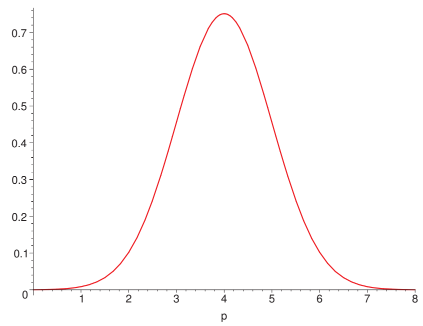
Wave packets: superpositions of plane waves with momenta in an interval of width σ_p around mom. p_0

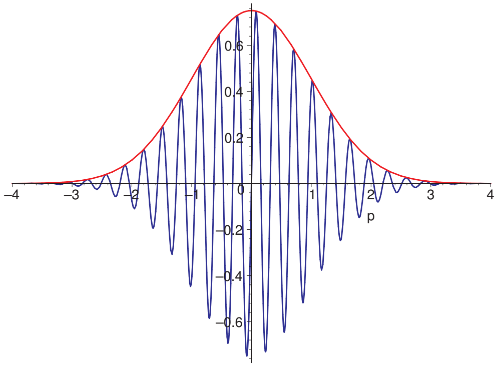
$$\sigma_x\,\sigma_p \geq 1/2$$ – QM uncertainty relation

W. packet centered at $\vec{x}_0 = 0$ at time $t = 0$:

$$\Psi(\vec{x};\,\vec{p}_0,\sigma_{\vec{p}}) \;=\; \int \frac{d^3p}{(2\pi)^3}\, f(\vec{p}-\vec{p}_0)\, e^{i\vec{p}\,\vec{x}}$$

Gaussian mom. space w. packet:





$$\sigma_x\sigma_p \;=\; 1/2$$ – minimum uncertainty packet

Propagating wave packets

Include time dependence:

$$\Psi(\vec{x},\,t) \;=\; \int \frac{d^3p}{(2\pi)^3}\, f(\vec{p}-\vec{p}_0)\, e^{i\vec{p}\vec{x}-iE(p)t}$$

Example: Gaussian wave packets

Momentum-space distribution:

$$f(\vec{p}-\vec{p}_0) \;=\; \frac{1}{(2\pi\sigma_p^2)^{3/4}}\exp\left\{-\frac{(\vec{p}-\vec{p}_0)^2}{4\sigma_p^2}\right\}$$

Coordinate-space wave packet for ν_i (neglecting spreading):

$$\diamondsuit \quad \boxed{\Psi_i(\vec{x},\,t) \;=\; e^{i\vec{p}_0\vec{x}-iE_i(p_0)t}\,\frac{1}{(2\pi\sigma_x^2)^{3/4}}\exp\left\{-\frac{(\vec{x}-\vec{v}_{gi}t)^2}{4\sigma_x^2}\right\}}\,, \qquad \sigma_x^2 = 1/(4\sigma_p^2)$$

QM wave packet approach

The evolved produced state:

$$|\nu_\alpha^{\rm fl}(\vec{x},t)\rangle \;=\; \sum_i U_{\alpha i}^*\,|\nu_i^{\rm mass}(\vec{x},t)\rangle \;=\; \sum_i U_{\alpha i}^*\,\Psi_i^P(\vec{x},t)|\nu_i^{\rm mass}\rangle$$

Transition amplitude:

$$\mathcal{A}_{\alpha\beta}(T,\vec{L}) \;=\; \langle\nu_\beta^{\rm fl}|\nu_\alpha^{\rm fl}(T,\vec{L})\rangle \;=\; \sum_i U_{\alpha i}^* U_{\beta i}\,\mathcal{A}_i(T,\vec{L})$$

Strongly suppressed unless $|\vec{L}-\vec{v}_{gi}T|\lesssim\sigma_x$. E.g., for Gaussian wave packets:

$$\mathcal{A}_i(T,\vec{L}) \,\propto\, \exp\left[-\frac{(\vec{L}-\vec{v}_{gi}T)^2}{4\sigma_x^2}\right], \qquad \sigma_x^2 \;\equiv\; \sigma_{xP}^2+\sigma_{xD}^2$$

Phase difference

Oscillations are due to phase differences of different mass eigenstates:

$$\Delta\phi = \Delta E \cdot T - \Delta p \cdot L \qquad \left(E_i = \sqrt{p_i^2 + m_i^2}\right)$$

For relativistic or quasi-degenerate neutrinos: $\Delta E \ll E, \quad \Delta p \ll p \quad \Rightarrow$

$$\Delta E = \frac{\partial E}{\partial p}\Delta p + \frac{\partial E}{\partial m^2}\Delta m^2 = v_g\, \Delta p + \frac{1}{2E}\Delta m^2$$

$$\Delta\phi = (v_g\, \Delta p + \frac{1}{2E}\Delta m^2)\, T - \Delta p \cdot L$$

$$= -\,(L - v_g\, T)\Delta p + \frac{\Delta m^2}{2E}\, T$$

In the center of wave packet $(L - v_g\, T) = 0\,!$ In general, $|L - v_g\, T| \lesssim \sigma_x$;

if $\sigma_x \Delta p \ll 1\,,\ (\Delta p \ll \sigma_p,\ \sigma_x \ll l_{\rm osc}) \quad \Rightarrow \quad |L - v_g\, T|\Delta p \ll 1 \quad \Rightarrow$

$$\Delta\phi \;=\; \frac{\Delta m^2}{2E}\,T\,, \qquad\qquad L \;\simeq\; v_g T \;\simeq T$$

– the result of the “same momentum” approach recovered!

$$\Delta\phi \;=\; \frac{\Delta m^2}{2E}\,T\,, \qquad\qquad L \;\simeq\; v_g T \;\simeq T$$

– the result of the “same momentum” approach recovered!

Now instead of expressing ΔE through Δp and Δm^2 express Δp through ΔE and Δm^2:

$$\Delta\phi = \frac{\Delta m^2}{2E}\,T\,, \qquad\qquad L \simeq v_g T \simeq T$$

– the result of the “same momentum” approach recovered!

Now instead of expressing ΔE through Δp and Δm^2 express Δp through ΔE and Δm^2:

♢ $$\Delta\phi = -\frac{1}{v_g}(L - v_g\,T)\Delta E + \frac{\Delta m^2}{2p}\,L \qquad \Rightarrow \qquad \frac{\Delta m^2}{2p}\,L$$

$$\Delta\phi = \frac{\Delta m^2}{2E}T\,, \qquad\qquad L \simeq v_g T \simeq T$$

– the result of the “same momentum” approach recovered!

Now instead of expressing ΔE through Δp and Δm^2 express Δp through ΔE and Δm^2:

$$\diamondsuit \qquad \Delta\phi = -\frac{1}{v_g}(L - v_g\,T)\Delta E + \frac{\Delta m^2}{2p}L \qquad \Rightarrow \qquad \frac{\Delta m^2}{2p}L$$

– the result of the “same energy” approach recovered!

$$\Delta\phi = \frac{\Delta m^2}{2E}T\,, \qquad L \simeq v_g T \simeq T$$

– the result of the “same momentum” approach recovered!

Now instead of expressing ΔE through Δp and Δm^2 express Δp through ΔE and Δm^2:

$$\diamondsuit \qquad \Delta\phi = -\frac{1}{v_g}(L - v_g\,T)\Delta E + \frac{\Delta m^2}{2p}L \qquad \Rightarrow \qquad \frac{\Delta m^2}{2p}L$$

– the result of the “same energy” approach recovered!

<u>The reasons why wrong assumptions give the correct result:</u>

$$\Delta\phi = \frac{\Delta m^2}{2E}\,T\,, \qquad L \simeq v_g T \simeq T$$

– the result of the “same momentum” approach recovered!

Now instead of expressing ΔE through Δp and Δm^2 express Δp through ΔE and Δm^2:

$$\diamondsuit \qquad \Delta\phi = -\frac{1}{v_g}(L - v_g\,T)\Delta E + \frac{\Delta m^2}{2p}\,L \qquad \Rightarrow \qquad \frac{\Delta m^2}{2p}\,L$$

– the result of the “same energy” approach recovered!

The reasons why wrong assumptions give the correct result:

- Neutrinos are relativistic or quasi-degenerate with $\Delta E \ll E$, $\Delta p \ll p$

$$\Delta\phi = \frac{\Delta m^2}{2E}\,T\,, \qquad L \simeq v_g T \simeq T$$

– the result of the "same momentum" approach recovered!

Now instead of expressing ΔE through Δp and Δm^2 express Δp through ΔE and Δm^2:

$$\diamondsuit \qquad \Delta\phi = -\frac{1}{v_g}(L - v_g T)\Delta E + \frac{\Delta m^2}{2p}\,L \qquad \Rightarrow \qquad \frac{\Delta m^2}{2p}\,L$$

– the result of the "same energy" approach recovered!

The reasons why wrong assumptions give the correct result:

- Neutrinos are relativistic or quasi-degenerate with $\Delta E \ll E$, $\Delta p \ll p$
- The size of the neutrino wave packet is small compared to the oscillation length: $\sigma_x \ll l_{\rm osc}$ (more precisely: energy uncertainty $\sigma_E \gg \Delta E$)

When are neutrino oscillations observable?

Keyword: *Coherence*

Neutrino flavour eigenstates ν_e, ν_μ and ν_τ are coherent superpositions of mass eigenstates ν_1, ν_2 and ν_3 $\Rightarrow$ oscillations are only observable if

- neutrino production and detection are coherent
- coherence is not (irreversibly) lost during neutrino propagation.

Possible decoherence at production (detection): If by accurate E and p measurements one can tell (through $E = \sqrt{p^2 + m^2}$) which mass eigenstate is emitted, the coherence is lost and oscillations disappear!

Full analogy with electron interference in double slit experiments: if one can establish which slit the detected electron has passed through, the interference fringes are washed out.

When are neutrino oscillations observable?

Decoherence can be considered in either configuration or momentum space.

Intrinsic QM uncertainties σ_E and σ_p of neutrino energy and momentum: related to the uncertainties of the neutrino production/detection time and coordinate through QM uncertainty principles.

Coherence typically requires $\sigma_E \gg \Delta E,\ \ \sigma_p \gg \Delta p$. This prevents determination of neutrino mass.

As soon as σ_E and σ_p become sufficiently small to allow determination of the neutrino mass at neutrino production or detection $(\sigma_{m^2} < \Delta m^2)$, uncertainty in 4-coordinate of neutrino production/detection becomes larger than $l_{\rm osc}$ $\Rightarrow$ oscillations become unobservable (Kayser 1981)

$\Rightarrow$ connection between space-time and energy-momentum pictures of neutrino production or detection decoherence.

Configuration - space picture

Oscillation phase: $\phi_{osc} = \Delta E \cdot t - \Delta p \cdot x$.

Fluctuation of ϕ_{osc} due to uncertainty in 4-coordinate of neutrino production:

$$\delta\phi_{osc} = \Delta E \cdot \delta t - \Delta p \cdot \delta x \,,$$

δt and δx limited by the duration of the neutrino production process σ_t and its spatial extension σ_X: $\delta t \lesssim \sigma_t$, $|\delta x| \lesssim \sigma_X$.

Configuration - space picture

Oscillation phase: $\phi_{osc} = \Delta E \cdot t - \Delta p \cdot x$.

Fluctuation of ϕ_{osc} due to uncertainty in 4-coordinate of neutrino production:

$$\delta\phi_{osc} = \Delta E \cdot \delta t - \Delta p \cdot \delta x \,,$$

δt and δx limited by the duration of the neutrino production process σ_t and its spatial extension σ_X: $\delta t \lesssim \sigma_t$, $|\delta x| \lesssim \sigma_X$.

For oscillations to be observable $\delta\phi_{osc}$ must be small – otherwise oscillations will be washed out upon averaging over $(t_P,\, x_P)$ $\Rightarrow$

$$|\Delta E \cdot \delta t - \Delta p \cdot \delta x| \ll 1$$

Barring accidental cancellations: $\Delta E \cdot \delta t \ll 1$, $\Delta p \cdot \delta x \ll 1$.

Configuration - space picture

Oscillation phase: $\phi_{osc} = \Delta E \cdot t - \Delta p \cdot x$.

Fluctuation of ϕ_{osc} due to uncertainty in 4-coordinate of neutrino production:

$$\delta\phi_{osc} = \Delta E \cdot \delta t - \Delta p \cdot \delta x\,,$$

δt and δx limited by the duration of the neutrino production process σ_t and its spatial extension σ_X: $\delta t \lesssim \sigma_t$, $|\delta x| \lesssim \sigma_X$.

For oscillations to be observable $\delta\phi_{osc}$ must be small – otherwise oscillations will be washed out upon averaging over (t_P, x_P) $\Rightarrow$

$$|\Delta E \cdot \delta t - \Delta p \cdot \delta x| \ll 1$$

Barring accidental cancellations: $\Delta E \cdot \delta t \ll 1$, $\Delta p \cdot \delta x \ll 1$. From

$$\delta t \lesssim \sigma_t \sim \sigma_E^{-1}, \qquad \delta x \lesssim \sigma_X \sim \sigma_p^{-1} \qquad \Rightarrow$$

$$\diamondsuit \qquad \Delta E \ll \sigma_E, \qquad \Delta p \ll \sigma_p\,.$$

Wave packet separation

Wave packets representing different mass eigenstate components have different group velocities v_{gi} $\Rightarrow$ after time $t_{\rm coh}$ (coherence time) they separate $\Rightarrow$ Neutrinos stop oscillating! (Only averaged effect observable).

Coherence time and length:

$$\Delta v \cdot t_{\rm coh} \simeq \sigma_x\,; \qquad\qquad l_{\rm coh} \simeq v t_{\rm coh}$$

$$\Delta v = \frac{p_i}{E_i} - \frac{p_k}{E_k} \simeq \frac{\Delta m^2}{2E^2}$$

$$\boxed{l_{\rm coh} \simeq \frac{v}{\Delta v}\sigma_x = \frac{2E^2}{\Delta m^2}\, v\sigma_x}$$

The standard formula for $P_{\rm osc}$ is obtained when the decoherence effects are negligible.

For Gaussian WPs:

Giunti, Kim & Lee, Phys. Lett. B274 (1992) 87:

$$\boxed{P_{\alpha\beta}(L,E)=\sum_{i,k}U_{\alpha i}U^*_{\beta i}U^*_{\alpha k}U_{\beta k}e^{-i(\Delta m^2_{ik}/2p)L}\;e^{-[L/(l_{\rm coh})_{ik}]^2-[\Delta E^2_{ik}/8\sigma_E^2]}}$$

$$(l_{\rm coh})_{ik}=2\sqrt{2}\frac{v_g}{|\Delta v_g|}\sigma_x=2\sqrt{2}\frac{2E^2}{|\Delta m^2_{ik}|}\sigma_x\,; \qquad \sigma_x=1/2\sigma_p=(1/2)(v_g/\sigma_E)$$

$$\frac{1}{\sigma_E^2}=\frac{1}{\sigma^2_{Eprod}}+\frac{1}{\sigma^2_{Edet}}$$

$$\Delta E_{ik}=\xi\frac{\Delta m^2_{ik}}{2E}$$

◇ Overall normalization obtained by imposing unitarity condition!

Are coherence constraints compatible?

Observability conditions for ν oscillations:

- Coherence of ν production and detection
- Coherence of ν propagation

Both conditions put upper limits on neutrino mass squared differences Δm^2 :

$$(1) \quad \Delta E_{jk} \sim \frac{\Delta m_{jk}^2}{2E} \ll \sigma_E; \qquad\qquad (2) \quad \frac{\Delta m_{jk}^2}{2E^2} L \ll \sigma_x \simeq v_g/\sigma_E$$

Are coherence constraints compatible?

Observability conditions for ν oscillations:

- Coherence of ν production and detection
- Coherence of ν propagation

Both conditions put upper limits on neutrino mass squared differences Δm^2 :

$$(1) \quad \Delta E_{jk} \sim \frac{\Delta m_{jk}^2}{2E} \ll \sigma_E; \qquad\qquad (2) \quad \frac{\Delta m_{jk}^2}{2E^2} L \ll \sigma_x \simeq v_g/\sigma_E$$

But: The constraints on σ_E work in opposite directions:

$$(1) \quad \Delta E_{jk} \sim \frac{\Delta m_{jk}^2}{2E} \ll \sigma_E \ll \frac{2E^2}{\Delta m_{jk}^2} \frac{v_g}{L} \quad (2)$$

Are coherence constraints compatible?

Observability conditions for ν oscillations:

- Coherence of ν production and detection
- Coherence of ν propagation

Both conditions put upper limits on neutrino mass squared differences Δm^2 :

$$(1)\quad \Delta E_{jk} \sim \frac{\Delta m^2_{jk}}{2E} \ll \sigma_E; \qquad\qquad (2)\quad \frac{\Delta m^2_{jk}}{2E^2} L \ll \sigma_x \simeq v_g/\sigma_E$$

But: The constraints on σ_E work in opposite directions:

$$(1)\quad \Delta E_{jk} \sim \frac{\Delta m^2_{jk}}{2E} \ll \sigma_E \ll \frac{2E^2}{\Delta m^2_{jk}} \frac{v_g}{L} \quad (2)$$

Are they compatible? – Yes, if LHS $\ll$ RHS $\Rightarrow$

$$\boxed{2\pi \frac{L}{l_{\rm osc}} \ll \frac{v_g}{\Delta v_g}\ (\gg 1)}$$

– fulfilled in all cases of practical interest

Are coherence conditions satisfied?

The coherence propagation condition: satisfied very well for all but astrophysical and cosmological neutrinos (solar, SN, relic ν's ...)

Are coherence conditions satisfied?

The coherence propagation condition: satisfied very well for all but astrophysical and cosmological neutrinos (solar, SN, relic ν's ...)

Coherent production/detection: usually satisfied extremely well due to the tininess of neutrino mass

Are coherence conditions satisfied?

The coherence propagation condition: satisfied very well for all but astrophysical and cosmological neutrinos (solar, SN, relic ν's ...)

Coherent production/detection: usually satisfied extremely well due to the tininess of neutrino mass

But: Is not automatically guaranteed in the case of “light” sterile neutrinos! $m_{sterile} \sim \mathrm{eV} - \mathrm{keV} - \mathrm{MeV}$ scale $\Rightarrow$ heavy compared to the “usual” (active) neutrinos

Are coherence conditions satisfied?

The coherence propagation condition: satisfied very well for all but astrophysical and cosmological neutrinos (solar, SN, relic ν's ...)

Coherent production/detection: usually satisfied extremely well due to the tininess of neutrino mass

But: Is not automatically guaranteed in the case of "light" sterile neutrinos! $m_{sterile} \sim \mathrm{eV} - \mathrm{keV} - \mathrm{MeV}$ scale $\Rightarrow$ heavy compared to the "usual" (active) neutrinos

Sterile neutrinos: hints from SBL accelerator experiments (LSND, MiniBooNE), reactor neutrino anomaly, keV sterile neutrinos, pulsar kicks, leptogenesis via ν oscillations, SN r-process nucleosynthesis, unconventional contributions to $2\beta 0\nu$ decay ...

Are coherence conditions satisfied?

The coherence propagation condition: satisfied very well for all but astrophysical and cosmological neutrinos (solar, SN, relic ν's ...)

Coherent production/detection: usually satisfied extremely well due to the tininess of neutrino mass

But: Is not automatically guaranteed in the case of “light” sterile neutrinos! $m_{sterile} \sim \mathrm{eV} - \mathrm{keV} - \mathrm{MeV}$ scale $\Rightarrow$ heavy compared to the “usual” (active) neutrinos

Sterile neutrinos: hints from SBL accelerator experiments (LSND, MiniBooNE), reactor neutrino anomaly, keV sterile neutrinos, pulsar kicks, leptogenesis via ν oscillations, SN r-process nucleosynthesis, unconventional contributions to $2\beta0\nu$ decay ...

Production/detection coherence has to be re-checked – important implications for some neutrino experiments!

Neutrino oscillations: *Coherence at macroscopic distances –*
$L > 10{,}000$ km in atmospheric neutrino experiments !

Universal oscillation formula?

The complete process: production – propagation – detection: factorization

$$\Gamma_{\rm tot} = \Gamma_{\rm prod}\, P_{\rm prop}\, \sigma_{\rm det}$$

with a universal $P_{\rm prop}$ is only possible when all 3 processes are independent

In general not true, and production – propagation – detection should be considered as a single inseparable process!

To get the standard formula one assumes for the emitted and absorbed states

$$|\nu_a^{\rm fl}\rangle = \sum_i U_{ai}^* \,|\nu_i^{\rm mass}\rangle$$

The weights of the mass eigenstates are just U_{ai}^* – do not depend on the masses of ν_i $\Rightarrow$ only true when the phase space volumes at production and detection do not depend on the mass of ν_i.

$\Rightarrow$

Universal oscillation formula?

This is only true if the charact. energy E at production (and detection) is large compared to all m_i (relativistic neutrinos), or compared to all $|m_i - m_k|$ (quasi-degenerate neutrinos).

$\Rightarrow$ Neutrino oscillations can be described by a universal probability only when neutrinos are relativistic or quasi-degenerate

Also: degree of coherence of the propagating neutrino state depends on the coherence of the production and detection processes

$\Rightarrow$ The standard formula for the oscillation probability is only valid when all decoherence effects are negligible !

Oscillation probability in vacuum

Oscillation probability in vacuum

The standard formula for osc. probability is stubbornly robust.

Oscillation probability in vacuum

The standard formula for osc. probability is stubbornly robust.

Validity conditions:

Oscillation probability in vacuum

The standard formula for osc. probability is stubbornly robust.

Validity conditions:

- Neutrinos are ultra-relativistic or quasi-degenerate in mass

Oscillation probability in vacuum

The standard formula for osc. probability is stubbornly robust.

Validity conditions:

- Neutrinos are ultra-relativistic or quasi-degenerate in mass
- Coherence conditions for neutrino production, propagation and detection are satisfied.

Oscillation probability in vacuum

The standard formula for osc. probability is stubbornly robust.

Validity conditions:

- Neutrinos are ultra-relativistic or quasi-degenerate in mass
- Coherence conditions for neutrino production, propagation and detection are satisfied.

Gives also the correct result in the case of strong coherence violation (complete averaging regime).

Oscillation probability in vacuum

The standard formula for osc. probability is stubbornly robust.

Validity conditions:

- Neutrinos are ultra-relativistic or quasi-degenerate in mass
- Coherence conditions for neutrino production, propagation and detection are satisfied.

Gives also the correct result in the case of strong coherence violation (complete averaging regime).

Gives only order of magnitude estimate when decoherence parameters are of order one.

Oscillation probability in vacuum

The standard formula for osc. probability is stubbornly robust.

Validity conditions:

- Neutrinos are ultra-relativistic or quasi-degenerate in mass
- Coherence conditions for neutrino production, propagation and detection are satisfied.

Gives also the correct result in the case of strong coherence violation (complete averaging regime).

Gives only order of magnitude estimate when decoherence parameters are of order one.

<u>But:</u> Conditions for partial decoherence are difficult to realize

Oscillation probability in vacuum

The standard formula for osc. probability is stubbornly robust.

Validity conditions:

- Neutrinos are ultra-relativistic or quasi-degenerate in mass
- Coherence conditions for neutrino production, propagation and detection are satisfied.

Gives also the correct result in the case of strong coherence violation (complete averaging regime).

Gives only order of magnitude estimate when decoherence parameters are of order one.

But: Conditions for partial decoherence are difficult to realize

They may still be realized if relatively heavy sterile neutrinos exist

Shortcomings of the QM w. packet approach

- Neutrino wave packet postulated rather than derived, widths estimated
- Production and detection processes are not (adequately) considered
- Inadequate normalization procedure. Normalization "by hand" is unavoidable.

Advantage: simplicity

QM vs QFT

- QM and QFT wave packet formalisms provide consistent approaches to neutrino oscillations.

QM vs QFT

- QM and QFT wave packet formalisms provide consistent approaches to neutrino oscillations.
- QFT approach is superior to the QM one:

QM vs QFT

- QM and QFT wave packet formalisms provide consistent approaches to neutrino oscillations.
- QFT approach is superior to the QM one:
  - Consistently takes into account neutrino production and detection mechanisms

QM vs QFT

- QM and QFT wave packet formalisms provide consistent approaches to neutrino oscillations.
- QFT approach is superior to the QM one:
  - Consistently takes into account neutrino production and detection mechanisms
  - Allows to obtain the neutrino wave packets used in the QM approach (instead of postulating them)

QM vs QFT

- QM and QFT wave packet formalisms provide consistent approaches to neutrino oscillations.
- QFT approach is superior to the QM one:
  - Consistently takes into account neutrino production and detection mechanisms
  - Allows to obtain the neutrino wave packets used in the QM approach (instead of postulating them)
  - Automatically produces correctly normalized oscillation probability and clarifies the normalization prescription of QM approach

QM vs QFT

- QM and QFT wave packet formalisms provide consistent approaches to neutrino oscillations.
- QFT approach is superior to the QM one:
  - Consistently takes into account neutrino production and detection mechanisms
  - Allows to obtain the neutrino wave packets used in the QM approach (instead of postulating them)
  - Automatically produces correctly normalized oscillation probability and clarifies the normalization prescription of QM approach
- $\Rightarrow$ the simplistic QM wave packet approach may need QFT-motivated modifications; however, once they have been done, one can still work within the QM framework without losing any essential physics content.

Instead of conclusion

Neutrinos have given (and still give) us a lot of surprises

Instead of conclusion

Neutrinos have given (and still give) us a lot of surprises

Many times in the past it appeared that the theory of neutrino oscillations was complete and finished but this has turned out to be wrong

Instead of conclusion

Neutrinos have given (and still give) us a lot of surprises

Many times in the past it appeared that the theory of neutrino oscillations was complete and finished but this has turned out to be wrong

Are we in the same situation now?

Backup slides

Oscillation probability in WP approach

Neutrino emission and detection times are not measured (or not accurately measured) in most experiments $\Rightarrow$ integration over T:

$$P(\nu_\alpha \to \nu_\beta; L) = \int dT\, P(\nu_\alpha \to \nu_\beta; T, L) = \sum_{i,k} U_{\alpha i}^* U_{\beta i} U_{\alpha k} U_{\beta k}^*\, e^{-i\frac{\Delta m_{ik}^2}{2\bar{P}}L}\, \tilde{I}_{ik}$$

Oscillation probability in WP approach

Neutrino emission and detection times are not measured (or not accurately measured) in most experiments $\Rightarrow$ integration over T:

$$P(\nu_\alpha \to \nu_\beta; L) = \int dT\, P(\nu_\alpha \to \nu_\beta; T, L) = \sum_{i,k} U_{\alpha i}^* U_{\beta i} U_{\alpha k} U_{\beta k}^*\, e^{-i\frac{\Delta m_{ik}^2}{2P} L}\, \tilde{I}_{ik}$$

$$\boxed{\begin{aligned}\tilde{I}_{ik} = N \int \frac{dq}{2\pi} f_i^S(r_k q - \Delta E_{ik}/2v + P_i) f_i^{D*}(r_k q - \Delta E_{ik}/2v + P_i) \\ \times f_k^{S*}(r_i q + \Delta E_{ik}/2v + P_k) f_k^D(r_i q + \Delta E_{ik}/2v + P_k)\, e^{i\frac{\Delta v}{v} q L}\end{aligned}}$$

Here: $v \equiv \frac{v_i + v_k}{2}$, $\Delta v \equiv v_k - v_i$, $r_{i,k} \equiv \frac{v_{i,k}}{v}$, $N \equiv 1/[2E_i(P) 2E_k(P) v]$

Oscillation probability in WP approach

Neutrino emission and detection times are not measured (or not accurately measured) in most experiments $\Rightarrow$ integration over T:

$$P(\nu_\alpha \to \nu_\beta; L) = \int dT\, P(\nu_\alpha \to \nu_\beta; T, L) = \sum_{i,k} U_{\alpha i}^* U_{\beta i} U_{\alpha k} U_{\beta k}^*\, e^{-i\frac{\Delta m_{ik}^2}{2P}L}\, \tilde{I}_{ik}$$

$$\boxed{\begin{aligned}\tilde{I}_{ik} = N \int \frac{dq}{2\pi} f_i^S(r_k q - \Delta E_{ik}/2v + P_i) f_i^{D*}(r_k q - \Delta E_{ik}/2v + P_i) \\ \times f_k^{S*}(r_i q + \Delta E_{ik}/2v + P_k) f_k^D(r_i q + \Delta E_{ik}/2v + P_k)\, e^{i\frac{\Delta v}{v} qL}\end{aligned}}$$

Here: $v \equiv \frac{v_i + v_k}{2}$, $\Delta v \equiv v_k - v_i$, $r_{i,k} \equiv \frac{v_{i,k}}{v}$, $N \equiv 1/[2E_i(P)2E_k(P)v]$

- For $(\Delta v/v)\sigma_p L \ll 1$ (i.e. $L \ll l_{\rm coh} = (v/\Delta v)\sigma_x$) $\tilde{I}_{ik}$ is approximately independent of L; in the opposite case $\tilde{I}_{ik}$ is strongly suppressed

Oscillation probability in WP approach

Neutrino emission and detection times are not measured (or not accurately measured) in most experiments $\Rightarrow$ integration over T:

$$P(\nu_\alpha \to \nu_\beta; L) = \int dT\, P(\nu_\alpha \to \nu_\beta; T, L) = \sum_{i,k} U_{\alpha i}^* U_{\beta i} U_{\alpha k} U_{\beta k}^*\, e^{-i\frac{\Delta m_{ik}^2}{2P}L}\, \tilde{I}_{ik}$$

$$\boxed{\begin{aligned}\tilde{I}_{ik} = N \int \frac{dq}{2\pi} f_i^S(r_k q - \Delta E_{ik}/2v + P_i) f_i^{D*}(r_k q - \Delta E_{ik}/2v + P_i) \\ \times f_k^{S*}(r_i q + \Delta E_{ik}/2v + P_k) f_k^D(r_i q + \Delta E_{ik}/2v + P_k)\, e^{i\frac{\Delta v}{v}qL}\end{aligned}}$$

Here: $v \equiv \frac{v_i + v_k}{2}\,,\quad \Delta v \equiv v_k - v_i\,,\quad r_{i,k} \equiv \frac{v_{i,k}}{v}\,,\quad N \equiv 1/[2E_i(P) 2E_k(P) v]$

- For $(\Delta v/v)\sigma_p L \ll 1$ (i.e. $L \ll l_{\rm coh} = (v/\Delta v)\sigma_x$) $\tilde{I}_{ik}$ is approximately independent of L; in the opposite case $\tilde{I}_{ik}$ is strongly suppressed
- $\tilde{I}_{ik}$ is also strongly suppressed unless $\Delta E_{ik}/v \ll \sigma_p$, i.e. $\Delta E_{ik} \ll \sigma_E$ – coherent production/detection condition

A manifestation of neutrino coherence

Even non-observation of neutrino oscillations at distances $L \ll l_{\rm osc}$ is a consequence of and an evidence for coherence of neutrino emission and detection! Two-flavour example (e.g. for ν_e emission and detection):

$$A_{\rm prod/det}(\nu_1) \sim \cos\theta\,, \qquad A_{\rm prod/det}(\nu_2) \sim \sin\theta \qquad \Rightarrow$$

$$A(\nu_e \to \nu_e) = \sum_{i=1,2} A_{\rm prod}(\nu_i) A_{\rm det}(\nu_i) \sim \cos^2\theta + e^{-i\Delta\phi}\sin^2\theta$$

Phase difference $\Delta\phi$ vanishes at short L $\Rightarrow$

$$P(\nu_e \to \nu_e) = (\cos^2\theta + \sin^2\theta)^2 = 1$$

If ν_1 and ν_2 were emitted and absorbed incoherently) $\Rightarrow$ one would have to sum probabilities rather than amplitudes:

$$P(\nu_e \to \nu_e) \sim \sum_{i=1,2} |A_{\rm prod}(\nu_i)|^2 |A_{\rm det}(\nu_i)|^2 \sim \cos^4\theta + \sin^4\theta \; < \; 1$$

Propagating wave packets

Include time dependence:

$$\Psi(\vec{x},\,t) \;=\; \int \frac{d^3p}{(2\pi)^3}\, f(\vec{p}-\vec{p}_0)\, e^{i\vec{p}\vec{x}-iE(p)t}$$

Example: Gaussian wave packets

Momentum-space distribution:

$$f(\vec{p}-\vec{p}_0) \;=\; \frac{1}{(2\pi\sigma_p^2)^{3/4}} \exp\left\{-\frac{(\vec{p}-\vec{p}_0)^2}{4\sigma_p^2}\right\}$$

Momentum dispersion: $\langle \vec{p}^{\,2}\rangle - \langle \vec{p}\,\rangle^2 = \sigma_p^2$.

Coordinate-space wave packet (neglecting spreading):

$$\Psi(\vec{x},\,t) \;=\; e^{i\vec{p}_0\vec{x}-iE(p_0)t}\,\frac{1}{(2\pi\sigma_x^2)^{3/4}} \exp\left\{-\frac{(\vec{x}-\vec{v}_g t)^2}{4\sigma_x^2}\right\}, \qquad \sigma_x^2 = 1/(4\sigma_p^2)$$

$$\langle \vec{x}\,\rangle = \vec{v}_g t\,; \qquad \langle \vec{x}^{\,2}\rangle - \langle \vec{x}\,\rangle^2 = \sigma_x^2\,.$$

QM wave packet approach

The evolved produced state:

$$|\nu_\alpha^{\rm fl}(\vec{x},t)\rangle \;=\; \sum_i U_{\alpha i}^*\,|\nu_i^{\rm mass}(\vec{x},t)\rangle \;=\; \sum_i U_{\alpha i}^*\,\Psi_i^S(\vec{x},t)|\nu_i^{\rm mass}\rangle$$

The coordinate-space wave function of the ith mass eigenstate (w. packet):

$$\boxed{\Psi_i^S(\vec{x},t) \;=\; \int\frac{d^3p}{(2\pi)^3}\,f_i^S(\vec{p})\,e^{i\vec{p}\vec{x}-iE_i(p)t}}$$

Momentum distribution function $f_i^S(\vec{p})$: sharp maximum at $\vec{p}=\vec{P}$ (width of the peak $\sigma_{pP}\ll P$).

$$E_i(p) \;=\; E_i(P) \;+\; \left.\frac{\partial E_i(p)}{\partial\vec{p}}\right|_{\vec{P}}(\vec{p}-\vec{P}) \;+\; \frac{1}{2}\left.\frac{\partial^2 E_i(p)}{\partial\vec{p}^2}\right|_{\vec{p}_0}(\vec{p}-\vec{P})^2 \;+\; \ldots$$

$$\vec{v}_i \;=\; \frac{\partial E_i(p)}{\partial\vec{p}} \;=\; \frac{\vec{p}}{E_i}\,, \qquad\qquad \alpha \;\equiv\; \frac{\partial^2 E_i(p)}{\partial\vec{p}^2} \;=\; \frac{m_i^2}{E_i^2}$$

Evolved neutrino state

$$\boxed{\Psi_i^S(\vec{x},\,t)\;\simeq\;e^{-iE_i(P)t+i\vec{P}\vec{x}}\,g_i^S(\vec{x}-\vec{v}_i t)}\qquad(\alpha\;\to\;0)$$

$$g_i^S(\vec{x}-\vec{v}_i t)\;\equiv\;\int\frac{d^3q}{(2\pi)^3}\,f_i^S(\vec{q}+\vec{P})\,e^{i\vec{q}(\vec{x}-\vec{v}_g t)}$$

Center of the wave packet: $\vec{x}-\vec{v}_i t=0$. Spatial length: $\sigma_{xP}\;\sim\;1/\sigma_{pP}$ (g_i^S decreases quickly for $|\vec{x}-\vec{v}_i t|\gtrsim\sigma_{xP}$).

Detected state (centered at $\vec{x}=\vec{L}$):

$$|\nu_\beta^{\rm fl}(\vec{x})\rangle\;=\;\sum_k U_{\beta k}^*\,\Psi_k^D(\vec{x})|\nu_i^{\rm mass}\rangle$$

The coordinate-space wave function of the ith mass eigenstate (w. packet):

$$\boxed{\Psi_i^D(\vec{x})\;=\;\int\frac{d^3p}{(2\pi)^3}\,f_i^D(\vec{p})\,e^{i\vec{p}(\vec{x}-\vec{L})}}$$

Oscillation probability

Transition amplitude:

$$\mathcal{A}_{\alpha\beta}(T,\vec{L}) \;=\; \langle\nu_\beta^{\rm fl}|\nu_\alpha^{\rm fl}(T,\vec{L})\rangle \;=\; \sum_i U_{\alpha i}^* U_{\beta i}\,\mathcal{A}_i(T,\vec{L})$$

$$\boxed{\mathcal{A}_i(T,\vec{L}) \;=\; \int\frac{d^3p}{(2\pi)^3}\, f_i^S(\vec{p})\, f_i^{D*}(\vec{p})\, e^{-iE_i(p)T+i\vec{p}\vec{L}}}$$

Strongly suppressed unless $|\vec{L}-\vec{v}_i T| \lesssim \sigma_x$. E.g., for Gaussian wave packets:

$$\mathcal{A}_i(T,\vec{L}) \;\propto\; \exp\left[-\frac{(\vec{L}-\vec{v}_i T)^2}{4\sigma_x^2}\right], \qquad \sigma_x^2 \;\equiv\; \sigma_{xP}^2+\sigma_{xD}^2$$

Oscillation probability:

$\diamondsuit$ $$P(\nu_\alpha\to\nu_\beta;T,\vec{L}) \;=\; |\mathcal{A}_{\alpha\beta}|^2 = \sum_{i,k} U_{\alpha i}^* U_{\beta i} U_{\alpha k} U_{\beta k}^*\,\mathcal{A}_i(T,\vec{L})\mathcal{A}_k^*(T,\vec{L})$$

Oscillation probability

Transition amplitude:

$$\mathcal{A}_{\alpha\beta}(T,\vec{L}) \;=\; \langle\nu_\beta^{\rm fl}|\nu_\alpha^{\rm fl}(T,\vec{L})\rangle \;=\; \sum_i U_{\alpha i}^* U_{\beta i}\,\mathcal{A}_i(T,\vec{L})$$

$$\boxed{\mathcal{A}_i(T,\vec{L}) \;=\; \int\frac{d^3p}{(2\pi)^3}\,f_i^S(\vec{p})\,f_i^{D*}(\vec{p})\,e^{-iE_i(p)T+i\vec{p}\vec{L}}}$$

Strongly suppressed unless $|\vec{L}-\vec{v}_iT|\lesssim\sigma_x$. E.g., for Gaussian wave packets:

$$\mathcal{A}_i(T,\vec{L}) \;\propto\; \exp\left[-\frac{(\vec{L}-\vec{v}_iT)^2}{4\sigma_x^2}\right], \qquad \sigma_x^2 \;\equiv\; \sigma_{xP}^2+\sigma_{xD}^2$$

Oscillation probability:

$\diamondsuit$ $$P(\nu_\alpha\to\nu_\beta;T,\vec{L}) \;=\; |\mathcal{A}_{\alpha\beta}|^2 = \sum_{i,k} U_{\alpha i}^* U_{\beta i} U_{\alpha k} U_{\beta k}^*\,\mathcal{A}_i(T,\vec{L})\mathcal{A}_k^*(T,\vec{L})$$

Configuration - space picture

Oscillation phase acquired over the distance x and time t:

$$\phi_{osc} = \Delta E \cdot t - \Delta p \cdot x\,.$$

Fluctuation of ϕ_{osc} due to uncertainty in 4-coordinate of neutrino production:

$$\delta\phi_{osc} = \Delta E \cdot \delta t - \Delta p \cdot \delta x\,,$$

δt and δx limited by the duration of the neutrino production process σ_t and its spatial extension σ_X: $\delta t \lesssim \sigma_t$, $|\delta x| \lesssim \sigma_X$.

Configuration - space picture

Oscillation phase acquired over the distance x and time t:

$$\phi_{osc} = \Delta E \cdot t - \Delta p \cdot x\,.$$

Fluctuation of ϕ_{osc} due to uncertainty in 4-coordinate of neutrino production:

$$\delta\phi_{osc} = \Delta E \cdot \delta t - \Delta p \cdot \delta x\,,$$

δt and δx limited by the duration of the neutrino production process σ_t and its spatial extension σ_X: $\delta t \lesssim \sigma_t$, $|\delta x| \lesssim \sigma_X$.

For oscillations to be observable $\delta\phi_{osc}$ must be small – otherwise oscillations will be washed out upon averaging over $(t_P,\, x_P)$ $\Rightarrow$

$$|\Delta E \cdot \delta t - \Delta p \cdot \delta x| \ll 1$$

Configuration - space picture

Oscillation phase acquired over the distance x and time t:

$$\phi_{osc} = \Delta E \cdot t - \Delta p \cdot x\,.$$

Fluctuation of ϕ_{osc} due to uncertainty in 4-coordinate of neutrino production:

$$\delta\phi_{osc} = \Delta E \cdot \delta t - \Delta p \cdot \delta x\,,$$

δt and δx limited by the duration of the neutrino production process σ_t and its spatial extension σ_X: $\delta t \lesssim \sigma_t$, $|\delta x| \lesssim \sigma_X$.

For oscillations to be observable $\delta\phi_{osc}$ must be small – otherwise oscillations will be washed out upon averaging over $(t_P,\, x_P)$ $\Rightarrow$

$$|\Delta E \cdot \delta t - \Delta p \cdot \delta x| \ll 1$$

Barring accidental cancellations: $\Delta E \cdot \delta t \ll 1$, $\Delta p \cdot \delta x \ll 1$. From

$$\delta t \lesssim \sigma_t \sim \sigma_E^{-1}, \qquad \delta x \lesssim \sigma_X \sim \sigma_p^{-1} \qquad \Rightarrow$$

$$\diamondsuit \quad \Delta E \ll \sigma_E, \qquad \Delta p \ll \sigma_p\,.$$

When are neutrino oscillations observable?

Conditions

$$\Delta E/\sigma_E \ll 1, \qquad \Delta p/\sigma_p \ll 1$$

are not Lorentz invariant! Can serve as coherent production conditions only when the neutrino source is at rest or non-relativistic – caution advised!

For non-relativistic neutrinos the condition $\Delta E/\sigma_E \ll 1$ is always violated $\Rightarrow$ no oscillations occur when neutrinos are non-relativistic in the ref. frame where their source is at rest or slowly moving.

But: for the usual neutrino oscillations one can always go to a Lorentz frame where one of the mass-eigenstate neutrinos is at rest!

Resolution of the apparent paradox: In this case the two terms in $\Delta E \cdot \delta t - \Delta p \cdot \delta x$ are no longer unrelated and nearly cancel each other.

$\Rightarrow \quad |\delta\phi_{osc}| \ll 1$ does not lead to $\Delta E/\sigma_E \ll 1,\ \Delta p/\sigma_p \ll 1$.

The standard osc. probability?

The standard formula for the oscillation probability corresponds to $\tilde{I}_{ik} = 1$.

Normaliz. condition:

$$\int \frac{d^3p}{(2\pi)^3} |f_i^S(\vec{p})|^2 |f_i^D(\vec{p})|^2 = 1$$

The normalization prescription

Oscillation probability calculated in QM w. packet approach is not automatically normalized ! Can be normalized "by hand" by imposing the unitarity condition:

$$\sum_\beta P_{\alpha\beta}(L) \;=\; 1\,.$$

This gives

$$\int dT |\mathcal{A}_i(L,T)|^2 = 1 \quad \Rightarrow \quad \tilde{I}_{ii} = N_1 \int \frac{dp}{2\pi v} |f_i^S(p)|^2\, |f_i^D(p)|^2 \;=\; 1$$

– important for proving Lorentz invariance of the oscillation probability.

Depends on the overlap of $f_i^S(p)$ and $f_i^S(p)$ $\Rightarrow$ no independent normalization of the produced and detected neutrino wave function would do!

In QFT approach the correctly normalized $P_{\alpha\beta}(L)$ is automatically obtained and the meaning of the normalization procedure adopted in the w. packet approach clarified

Lorentz invariance of oscillation probability

1. "Paradox" of neutrino w. packet length

For neutrino production in decays of unstable particles at rest (e.g. $\pi \to \mu\nu_\mu$):

$$\sigma_E \simeq \tau^{-1} = \Gamma_\pi\,, \qquad\qquad \sigma_x \simeq \frac{v_g}{\sigma_E} \simeq \frac{v_g}{\Gamma_\pi} \;\; (= v_g\tau)$$

Lorentz invariance of oscillation probability

1. "Paradox" of neutrino w. packet length

For neutrino production in decays of unstable particles at rest (e.g. $\pi \to \mu\nu_\mu$):

$$\sigma_E \simeq \tau^{-1} = \Gamma_\pi\,, \qquad\qquad \sigma_x \simeq \frac{v_g}{\sigma_E} \simeq \frac{v_g}{\Gamma_\pi} \;(= v_g\tau)$$

For decay in flight: $\Gamma'_\pi = (m_\pi/E_\pi)\Gamma_\pi$. One might expect

$$\sigma'_x \simeq \frac{E_\pi}{m_\pi}\sigma_x > \sigma_x\,.$$

Lorentz invariance of oscillation probability

1. "Paradox" of neutrino w. packet length

For neutrino production in decays of unstable particles at rest (e.g. $\pi \to \mu\nu_\mu$):

$$\sigma_E \simeq \tau^{-1} = \Gamma_\pi\,, \qquad\qquad \sigma_x \simeq \frac{v_g}{\sigma_E} \simeq \frac{v_g}{\Gamma_\pi} \;(= v_g\tau)$$

For decay in flight: $\Gamma'_\pi = (m_\pi/E_\pi)\Gamma_\pi$. One might expect

$$\sigma'_x \simeq \frac{E_\pi}{m_\pi}\sigma_x > \sigma_x\,.$$

On the other hand, if the decaying pion is boosted in the direction of the neutrino momentum, the neutrino w. packet should be Lorentz-contracted !

Lorentz invariance of oscillation probability

1. "Paradox" of neutrino w. packet length

For neutrino production in decays of unstable particles at rest (e.g. $\pi \to \mu\nu_\mu$):

$$\sigma_E \simeq \tau^{-1} = \Gamma_\pi\,, \qquad\qquad \sigma_x \simeq \frac{v_g}{\sigma_E} \simeq \frac{v_g}{\Gamma_\pi} \;\; (= v_g\tau)$$

For decay in flight: $\Gamma'_\pi = (m_\pi/E_\pi)\Gamma_\pi$. One might expect

$$\sigma'_x \simeq \frac{E_\pi}{m_\pi}\sigma_x > \sigma_x\,.$$

On the other hand, if the decaying pion is boosted in the direction of the neutrino momentum, the neutrino w. packet should be Lorentz-contracted !

<u>The solution:</u> pion decay takes finite time. During the decay time the pion moves over distance $l = u\tau'$ ("chases" the neutrino if $u > 0$).

$$\sigma'_x \simeq v'_g/\Gamma' - l = v'_g\tau' - u\tau' = (v'_g - u)\gamma_u\tau = \frac{v_g\tau}{\gamma_u(1+v_g u)}\,,$$

[the relativ. law of addition of velocities: $v'_g = (v_g+u)/(1+v_g u)$].

Lorentz invariance issues – contd.

That is

$$\sigma'_x = \frac{\sigma_x}{\gamma_u(1+v_g u)}$$

For relativistic neutrinos $v_g \approx v'_g \approx 1 \quad \Rightarrow$

$$\sigma'_x = \sigma_x \sqrt{\frac{1-u}{1+u}}$$

$\Rightarrow$ when the pion is boosted in the direction of neutrino emission $(u > 0)$ the neutrino wave packet gets contracted; when it is boosted in the opposite direction $(u < 0)$ – the wave packet gets dilated.

Lorentz invariance issues – contd.

The oscillation probability must be Lorentz invariant ! But: L. invariance is not obvious in QM w. packet approach which (unlike QFT) is not manifestly Lorentz covariant.

Lorentz invariance issues – contd.

The oscillation probability must be Lorentz invariant ! But: L. invariance is not obvious in QM w. packet approach which (unlike QFT) is not manifestly Lorentz covariant.

How can we see Lorentz invariance of the standard formula for the oscillation probability ? P_{ab} depends on L/p (contains factors $\exp[-i\frac{\Delta m_{ik}^2}{2p}L]$). Is L/p Lorentz invariant?

Lorentz invariance issues – contd.

The oscillation probability must be Lorentz invariant ! But: L. invariance is not obvious in QM w. packet approach which (unlike QFT) is not manifestly Lorentz covariant.

How can we see Lorentz invariance of the standard formula for the oscillation probability ? P_{ab} depends on L/p (contains factors $\exp[-i\frac{\Delta m_{ik}^2}{2p}L]$). Is L/p Lorentz invariant? Lorentz transformations:

$$L' = \gamma_u(L + ut)\,, \qquad t' = \gamma_u(t + uL)\,,$$

$$E' = \gamma_u(E + up)\,, \qquad p' = \gamma_u(p + uE)\,.$$

Lorentz invariance issues – contd.

The oscillation probability must be Lorentz invariant ! But: L. invariance is not obvious in QM w. packet approach which (unlike QFT) is not manifestly Lorentz covariant.

How can we see Lorentz invariance of the standard formula for the oscillation probability ? P_{ab} depends on L/p (contains factors $\exp[-i\frac{\Delta m_{ik}^2}{2p}L]$). Is L/p Lorentz invariant? Lorentz transformations:

$$L' = \gamma_u(L + ut)\,, \qquad t' = \gamma_u(t + uL)\,,$$
$$E' = \gamma_u(E + up)\,, \qquad p' = \gamma_u(p + uE)\,.$$

The stand. osc. formula results when (i) production and detection and (ii) propagation are coherent; for neutrinos from conventional sources (i) implies $\sigma_x \ll l_{\rm osc}$ $\Rightarrow$ one can consider neutrinos pointlike and set $L = v_g t$.
$\Rightarrow$ $L' = \gamma_u L(1 + u/v_g)$.

Lorentz invariance issues – contd.

The oscillation probability must be Lorentz invariant ! But: L. invariance is not obvious in QM w. packet approach which (unlike QFT) is not manifestly Lorentz covariant.

How can we see Lorentz invariance of the standard formula for the oscillation probability ? P_{ab} depends on L/p (contains factors $\exp[-i\frac{\Delta m_{ik}^2}{2p}L]$). Is L/p Lorentz invariant? Lorentz transformations:

$$L' = \gamma_u(L + ut)\,, \qquad t' = \gamma_u(t + uL)\,,$$
$$E' = \gamma_u(E + up)\,, \qquad p' = \gamma_u(p + uE)\,.$$

The stand. osc. formula results when (i) production and detection and (ii) propagation are coherent; for neutrinos from conventional sources (i) implies $\sigma_x \ll l_{\rm osc}$ $\Rightarrow$ one can consider neutrinos pointlike and set $L = v_g t$.

$\Rightarrow$ $L' = \gamma_u L(1 + u/v_g)$. On the other hand: $v_g = p/E$

$\Rightarrow$ $p' = \gamma_u p(1 + u/v_g)$.

Lorentz invariance issues – contd.

The oscillation probability must be Lorentz invariant ! But: L. invariance is not obvious in QM w. packet approach which (unlike QFT) is not manifestly Lorentz covariant.

How can we see Lorentz invariance of the standard formula for the oscillation probability ? P_{ab} depends on L/p (contains factors $\exp[-i\frac{\Delta m_{ik}^2}{2p}L]$). Is L/p Lorentz invariant? Lorentz transformations:

$$L' = \gamma_u(L + ut)\,, \qquad t' = \gamma_u(t + uL)\,,$$

$$E' = \gamma_u(E + up)\,, \qquad p' = \gamma_u(p + uE)\,.$$

The stand. osc. formula results when (i) production and detection and (ii) propagation are coherent; for neutrinos from conventional sources (i) implies $\sigma_x \ll l_{\rm osc}$ $\Rightarrow$ one can consider neutrinos pointlike and set $L = v_g t$.

$\Rightarrow$ $L' = \gamma_u L(1 + u/v_g)$. On the other hand: $v_g = p/E$

$\Rightarrow$ $p' = \gamma_u p(1 + u/v_g)$.

$\Rightarrow$ $$\boxed{L'/p' \,=\, L/p}$$

Lorentz invariance issues – contd.

A more general argument (applies also to Mössbauer neutrinos which are not pointlike): Consider the phase difference

$$\diamondsuit \qquad \Delta\phi = -\frac{1}{v_g}(L - v_g\, t)\Delta E + \frac{\Delta m^2}{2p} L$$

– a Lorentz invariant quantity, though the two terms are in not in general separately Lorentz invariant.

But: If the 1st term is negligible in all Lorentz frames, the second term is Lorentz invariant by itself $\Rightarrow$ L/p is Lorentz invariant.

The 1st term can be neglected when the production/detection coherence conditions are satisfied. In particular, it vanishes in the limit of pointlike neutrinos $L = v_g t$. N.B.:

$$L' - v_g' t' = \gamma_u \left[(L + ut) - \frac{v_g + u}{1 + v_g u}(t + uL)\right] = \frac{L - v_g t}{\gamma_u (1 + v_g u)}\,,$$

i.e. the condition $L = v_g t$ is Lorentz invariant. MB neutrinos: $\Delta E \simeq 0$.

Lorentz invariance issues – contd.

The oscillation probability must be Lorentz invariant even when the coherence conditions are not satisfied !
Lorentz invariance is enforced by the normalization condition.

$$P_{ab}(L) \;=\; \sum_{i,k} U_{ai} U_{bi}^* U_{ak}^* U_{bk}\, I_{ik}(L)\,, \qquad \text{where}$$

$$I_{ik}(L) \;\equiv\; \int dT\, \mathcal{A}_i(L,T) \mathcal{A}_k^*(L,T) e^{-i\Delta\phi_{ik}}$$

From the norm. cond. $\int dT\, |\mathcal{A}_i(L,T)|^2 \;=\; 1 \qquad \Rightarrow$

$$|\mathcal{A}_i|^2 dT \;=\; inv. \quad \Rightarrow \quad |\mathcal{A}_i||\mathcal{A}_k| dT \;=\; inv. \quad \Rightarrow \quad \mathcal{A}_i \mathcal{A}_k^* dT \;=\; inv.$$

The phase difference $\Delta\phi_{ik} = \Delta E_{ik} T - \Delta p_{ik} L$ is also Lorentz invariant $\Rightarrow$
so is $I_{ik}(L)$, and consequently $P_{ab}(L)$.

Longitudinal *vs.* transversal w.p. dispersion

Spreading of the wave packets: consequence of the fact that the there is a spread of momenta inside of the wave packets and of the p-dependence of the group velocity.

$$v_{spr}^{i} \simeq \frac{\partial v_i}{\partial p^j}\sigma_p^j = \frac{1}{E}(\delta_{ij} - v_i v_j)\sigma_p^j = \frac{1}{E}[\sigma_p^i - v_i(\vec{v}\vec{\sigma_p})]$$

This gives

$$v_{spr.}^{\perp} = \frac{\sigma_p}{E}\,, \qquad v_{spr.}^{||} = \frac{\sigma_p}{E}(1 - v^2) = \frac{\sigma_p}{E}\frac{m^2}{E^2}$$

$t_{transv} \sim E/\sigma_p^2, \;\; t_{long.} \sim E^3/\sigma_p^2 m^2$.

Coherence production conditions

Coherence production conditions:

$$|\Delta E| \ll \sigma_E\,, \qquad\qquad |\Delta p| \ll \sigma_p\,.$$

On the other hand:

$$\Delta E \simeq v_g \Delta p + \frac{\Delta m^2}{2E}.$$

Constraint $|\Delta E| \ll \sigma_E \quad \Rightarrow$

$$\left|\frac{v_g \Delta p}{\sigma_E} + \frac{\Delta m^2}{2E\sigma_E}\right| \ll 1. \qquad\qquad (*)$$

(a) The two terms in ΔE do not approximately cancel each other. $\Rightarrow$
$v_g|\Delta p| \ll \sigma_E \le \sigma_p$, i.e. for relativistic neutrinos $|\Delta p| \ll \sigma_p$ follows from $|\Delta E| \ll \sigma_E$.

(b1) There is a strong cancellation, but both terms on the l.h.s. of (*) are small – see case (a).

(b2) Strong cancellation, but both terms on the l.h.s. of (*) are $\gtrsim 1$: momentum condition is independent. But: the only known case – Mössbauer neutrinos.

Do charged leptons oscillate?

Do charged leptons oscillate?

What do we mean by charged leptons?
The usual $e^{\pm}$, $\mu^{\pm}$ and $\tau^{\pm}$ are mass eigenstates $\Rightarrow$ do not oscillate.

[Also: unlike neutrinos, they participate also in EM interactions (and are normally detected via these interactions) which are flavour-blind.]

Assume we create a muon at $t_0 = 0$ and $\vec{x}_0 = 0$. Neglecting muon decay, we have

$$|\Psi(0)\rangle = |\mu\rangle\,; \quad |\Psi(\vec{x}, t)\rangle = e^{-ip_\mu x}|\mu\rangle \quad \Rightarrow \quad P_{\mu\mu} = |\langle\mu|\Psi(\vec{x}\,, t)\rangle|^2 = 1$$

Assume now we manage to create a coherent superposition of μ and e:

$$|\Psi(0)\rangle = \cos\theta|\mu\rangle + \sin\theta|e\rangle$$

The weights of μ and e in the initial state: $\cos^2\theta$ and $\sin^2\theta$.

Do charged leptons oscillate?

Evolved state:

$$|\Psi(\vec{x},\, t)\rangle \;=\; e^{-ip_\mu x}\cos\theta|\mu\rangle \;+\; e^{-ip_e x}\sin\theta|e\rangle$$

The probabilities of finding μ and e:

$$\begin{aligned} P_\mu &= |\langle\mu|\Psi(\vec{x}\,,t)\rangle|^2 = |e^{-ip_\mu x}\cos\theta|^2 = \cos^2\theta \\ P_e &= |\langle e|\Psi(\vec{x}\,,t)\rangle|^2 = |e^{-ip_e x}\sin\theta|^2 = \sin^2\theta \end{aligned}$$

– are the same! $\Rightarrow$ There are no oscillations between mass eigenstates, no matter if the initial state is pure or (coherently) mixed

$$\Downarrow$$

There are no oscillations between e, μ and τ !

[NB: The same for neutrinos – initially produced ν_e can with some probability oscillate into ν_μ or ν_τ, but the weights of ν_1, ν_2 and ν_3 that were in the initial state will remain the same!]

Is that the full answer?

Can we imagine a situation when one creates a coherent superposition of e, μ and τ and then also detects their coherent superposition (the same or different) rather than individual mass-eigenstate charged leptons?

Charged - current weak interactions look completely symmetric w.r.t. neutrinos and charged leptons!

$$\mathcal{L}_{\rm CC} = -\frac{g}{\sqrt{2}}\,(\bar{e}_{aL}\gamma^{\mu}U_{ai}\nu_{iL})\,W_{\mu}^{-} + h.c.\,, \qquad U = V_L^{\dagger}V_{\nu}$$

Why do we say that charged leptons are emitted and detected in mass eigenstates and neutrinos in flavour states (superpositions of mass eigenstates) and not vice versa? Or not both as some superpositions of mass eigenstates? E.g.

$|e_1\rangle = U_{1e}|e\rangle + U_{1\mu}|\mu\rangle + U_{1\tau}|\tau\rangle$ is emitted or detected together with ν_1,
$|e_2\rangle = U_{2e}|e\rangle + U_{2\mu}|\mu\rangle + U_{2\tau}|\tau\rangle$ is emitted or detected together with ν_2,
$|e_3\rangle = U_{3e}|e\rangle + U_{3\mu}|\mu\rangle + U_{3\tau}|\tau\rangle$ is emitted or detected together with ν_3.

Why do neutrinos oscillate?

Because they are emitted (and absorbed) alongside charged leptons of definite mass $e^\pm$, $\mu^\pm$ or $\tau^\pm$. (This "measures" the flavour of neutrinos). How do we know that charged leptons are in mass eigenstates?

(1) Beta decay: only electrons are emitted together with neutrinos. Emission of $\mu^\pm$ and $\tau^\pm$ is forbidden by energy conservation.

(2) Decays $\pi^\pm \to \mu^\pm \nu$, $\pi^\pm \to e^\pm \nu$ (or $K^\pm \to \mu^\pm \nu$, $K^\pm \to e^\pm \nu$). Here emission of both muons and electrons is allowed.

Assume a coherent superposition of e and μ is produced in pion decay (nearly) at rest. The energy uncertainty of the charged lepton:

$$\sigma_E \simeq \Gamma_\pi = 2.5 \cdot 10^{-8}\ \mathrm{eV}$$

Uncertainty in the mass determination ($\sqrt{(2E\sigma_E)^2 + (2p\sigma_p)^2]} \simeq 2\sqrt{2}E\sigma_E$):

$$\sigma_{m^2} \sim 2\sqrt{2}E\sigma_E \simeq 2\sqrt{2} \cdot (90\ \mathrm{MeV}) \cdot (2.5 \cdot 10^{-8}\ \mathrm{eV}) \simeq 6.4\ \mathrm{eV}^2$$

Do charged leptons oscillate?

This has to be compared with $m_\mu^2 - m_e^2 \simeq (106\ \mathrm{MeV})^2 \quad \Rightarrow$
Different mass-eigenstate charged leptons are emitted incoherently!

This provides a “measurement” of the flavour of the emitted neutrino

For pion decay in flight: assume pion’s energy is E_0. The energies of the produced charged leptons are rescaled as $E \to E\,(E_0/m_\pi)$, but the pion decay width (and so σ_E) is rescaled as $\Gamma_\pi \to \Gamma_\pi(m_\pi/E_0) \quad \Rightarrow$
$[(2E\sigma_E)^2 + (2p\sigma_p)^2]^{1/2}$ remains the same (σ_{m^2} a Lorentz invariant quantity).

$$\Downarrow$$

- ♢ Charged leptons produced in $\pi^\pm \to l^\pm \nu$ and $K^\pm \to l^\pm \nu$ decays are always emitted as mass eigenstates and not as coherent superpositions of different mass eigenstates because of their very large Δm^2.

- ♢ Therefore even oscillations between e_1, e_1 and e_3 (or any other superpositions of e, μ and τ) are not possible.

Do charged leptons oscillate?

The masses and decay widths of $\pi^{\pm}$, $K^{\pm}$ are rather small $\Rightarrow$ σ_{m^2} small.
How about decays of $W^{\pm}$? For $W^{\pm} \to l^{\pm}\nu$ decays at rest:

$$\Gamma^0_{W\to l_a \nu} \simeq \frac{G_F m_W^3}{6\sqrt{2}\pi} \simeq 230 \ \mathrm{MeV}$$

$$\Rightarrow \qquad \sigma_{m^2} \ \sim \ 2\sqrt{2}\, E\sigma_E \ \simeq \ 2\sqrt{2}\cdot 40 \ \mathrm{GeV} \cdot 230 \ \mathrm{MeV} \ \simeq \ (5 \ \mathrm{GeV})^2 \,.$$

Thus

$$\sigma_{m^2} \gg m_\mu^2 - m_e^2 \,, \qquad \sigma_{m^2} > m_\tau^2 - m_\mu^2 \simeq (1.77 \ \mathrm{GeV})^2 \,,$$

$\Rightarrow$ all three charged leptons are produced *coherently* in $W^{\pm}$ decays.
Can one then observe oscillations between their different coh. superpositions?
Coherence length $l_{\mathrm{coh}} \simeq \sigma_x/\Delta v_g$:

$$(l_{\mathrm{coh}})_{\mathrm{max}} \simeq [\Gamma^0_{W\to l_a \nu}(\Delta v_g)_{\mathrm{min}}]^{-1} \simeq \frac{3\sqrt{2}\pi}{G_F m_W (m_\mu^2 - m_e^2)} \simeq 2.5 \times 10^{-8} \ \mathrm{cm}\,.$$

$\Rightarrow$ $l^{\pm}$ loose their coherence almost immediately after their production

$\Downarrow$

Do charged leptons oscillate?

What about $W^\pm \to l^\pm \nu$ decays in flight? Let γ be the Lorentz factor of $W^\pm$. $(\Delta v_g)_{\rm min} \simeq \Delta m^2_{\mu e}/2E^2 \equiv (m_\mu^2 - m_e^2)/2E^2$ and the partial decay width of $W^\pm$ scale with γ as

$$(\Delta v_g)_{\rm min} \to \gamma^{-2} (\Delta v_g)_{\rm min}\,, \qquad \Gamma^0_{W\to l_a \nu} \to \gamma^{-1} \Gamma^0_{W\to l_a \nu}\,.$$

Therefore the maximum coherence length $(l_{\rm coh})_{\rm max} \simeq \sigma_x/(\Delta v_g)_{\rm min} \simeq 1/[\Gamma^0_{W\to l_a\nu}(\Delta v_g)_{\rm min}]$ scales as

$$(l_{\rm coh})_{\rm max} \to \gamma^3 (l_{\rm coh})_{\rm max}\,.$$

In order for $(l_{\rm coh})_{\rm max}$ to be larger than e.g. 1 m, one would need $\gamma \gtrsim 1600$, or $E_W \gtrsim 130$ TeV – far above presently feasible energies.

Do charged leptons oscillate?

What about $W^{\pm} \to l^{\pm}\nu$ decays in flight? Let γ be the Lorentz factor of $W^{\pm}$. $(\Delta v_g)_{\rm min} \simeq \Delta m_{\mu e}^2/2E^2 \equiv (m_\mu^2 - m_e^2)/2E^2$ and the partial decay width of $W^{\pm}$ scale with γ as

$$(\Delta v_g)_{\rm min} \to \gamma^{-2}(\Delta v_g)_{\rm min}\,, \qquad \Gamma^0_{W\to l_a\nu} \to \gamma^{-1}\Gamma^0_{W\to l_a\nu}\,.$$

Therefore the maximum coherence length $(l_{\rm coh})_{\rm max} \simeq \sigma_x/(\Delta v_g)_{\rm min} \simeq 1/[\Gamma^0_{W\to l_a\nu}(\Delta v_g)_{\rm min}]$ scales as

$$(l_{\rm coh})_{\rm max} \to \gamma^3 (l_{\rm coh})_{\rm max}\,.$$

In order for $(l_{\rm coh})_{\rm max}$ to be larger than e.g. 1 m, one would need $\gamma \gtrsim 1600$, or $E_W \gtrsim 130$ TeV – far above presently feasible energies.

N.B.: Even if coherence was satisfied for charged leptons, to fix the composition of the mixed $l^{\pm}$ state in terms of e, μ and τ one would have to detect the accompanying neutrino as a state different from $\nu_{\rm fl}$ – e.g. as a mass eigenstate. Not possible within the standard model !

Extensions of the standard model?

Consider the SM amended by three heavy RH neutrinos N_i (seesaw model) plus an extra Higgs doublet. In this model N_i can decay into a charged lepton and charged Higgs boson:

$$N_i \to e_i^- + \Phi^+ .$$

Decays are caused by the Yukawa coupling Lagrangian

$$\mathcal{L}_Y = Y_{ai} \bar{L}_a N_{Ri} \Phi \; + \; h.c. \, ,$$

In the basis where the mass matrices of N_i and $l^\pm$ have been diagonalized, the Yukawa coupling matrix Y_{ai} is in general not diagonal $\Rightarrow$ in the decay of a mass-eigenstate sterile neutrino N_i any of the three charged leptons $e_a = e,\, \mu,\, \tau$ can be produced.

What are the conditions for the produced charged lepton state e_i to be a coherent superposition of the mass eigenstates e_a:

$$|e_i\rangle = [(Y^\dagger Y)_{ii}]^{-1/2} \sum_a Y_{ia}^\dagger \, |e_a\rangle \, ,$$

and how long this state can maintain its coherence?

Extensions of the standard model?

Neglecting the masses of $\Phi^\pm$ and $l^\pm$ compared to the mass M_i of the sterile neutrino:

$$\Gamma_i^0 \simeq \alpha_i M_i \,, \qquad \text{where} \qquad \alpha_i \equiv \frac{(Y^\dagger Y)_{ii}}{16\pi}\,.$$

Coherent production condition:

$$2\sqrt{2}\,E\,\Gamma_i^0 \simeq 2\sqrt{2}\,(M_i/2)\,\alpha_i M_i > \max\{m_\mu^2 - m_e^2,\; m_\tau^2 - m_\mu^2\}\,,$$

or

$$\alpha_i > 2.2\,(\mathrm{GeV}/M_i)^2\,.$$

From $l_{\rm coh} = \sigma_x v_g/\Delta v_g$ the coherence length for the emitted charged lepton state:

$$l_{\rm coh} \simeq \frac{M_i^2}{2\Gamma_i^0(m_\tau^2 - m_\mu^2)} \simeq 3.1\times 10^{-15}\,\alpha_i^{-1}\,\frac{M_i}{\mathrm{GeV}}\;\;\mathrm{cm}\,.$$

$\Rightarrow$

Extensions of the standard model?

$$l_{\rm coh} < 1.4 \times 10^{-15}\ {\rm cm}\ (M_i/{\rm GeV})^3\,.$$

For N_i decays in flight the r.h.s. has to be multiplied by γ^3 $\Rightarrow$ $(M_i/{\rm GeV})^3$ has to be replaced by $(E_i/{\rm GeV})^3$.

The charged lepton state will maintain its coherence over the distance $\sim 1\ m$ if

$$E_i \gtrsim 400\ {\rm TeV} \quad \Rightarrow \quad (Y^\dagger Y)_{ii} \gtrsim 1.3 \times 10^{-11}\,.$$

If only e and μ are to be produced coherently, a milder lower limit on E_i results:

$$E_i \gtrsim 10\ {\rm TeV}\,, \qquad (Y^\dagger Y)_{ii} \gtrsim 8.5 \times 10^{-11}\,.$$

Extensions of the standard model?

If the condition for coherent creation of the charged lepton state is satisfied and this state is detected through the inverse decay process before it loses its coherence, it may exhibit oscillations: a mass eigenstate sterile neutrino N_j different from N_i can be produced in the detection process $\Rightarrow$ the state e_i has oscillated into e_j.

Charged leptons would be able to oscillate, leading to a non-zero probability of the emission or absorption of a different sterile neutrino mass eigenstate N_j in the processes $e_j^{\pm} + \Phi^{\mp} \to N_j$ or $e_j^{\pm} + N_j \to \Phi^{\pm}$.

$\Rightarrow$ The roles of neutrinos and charged leptons reversed compared to the usual situation because of sterile neutrinos being much heavier than the charged leptons.

QFT approach to neutrino oscillations

Calc. from 1st principles – QFT approach

Production - propagation - detection treated as a single inseparable process.
External particles are described by wave packets, neutrinos – by propagators

One-particle states of external particles:

$$|A\rangle = \int [dp]\, f_A(\vec{p}, \vec{P})\, |A, \vec{p}\rangle\,, \qquad [dp] \equiv \frac{d^3p}{(2\pi)^3\sqrt{2E_A(\vec{p})}}$$

$|A, \vec{p}\rangle$ – one-particle momentum eigenstate corresponding to momentum $\vec{p}$ and energy $E_A(\vec{p})$ (free particles: $E_A(\vec{p}) = \sqrt{\vec{p}^2 + m_A^2}$). The normalization condition for the plane wave states $|A, \vec{p}\rangle$:

$$\langle A, \vec{p}'|A, \vec{p}\rangle = 2E_A(\vec{p})\,(2\pi)^3\delta^{(3)}(\vec{p} - \vec{p}')\,.$$

$f_A(\vec{p}, \vec{P})$ – momentum distribution function with the mean momentum $\vec{P}$.
Normalization condition: $\langle A|A\rangle = 1 \quad \Rightarrow \quad \int d^3p\, |f_A(\vec{p})|^2/(2\pi)^3 = 1$.

QFT approach – contd.

Coordinate-space wave packet with maximum at $\vec{x} = \vec{x}_0$ at the time $t - t_0$:

$$\Psi_A(x) = \int [dp]\, f_A(\vec{p}) e^{-iE_A(\vec{p})(t-t_0)+i\vec{p}(\vec{x}-\vec{x}_0)}$$

Consistent with the usual QFT definition of the wave function:

$$\Psi_A(x) = \langle 0|\hat{\Psi}_A(x)|A\rangle\,.$$

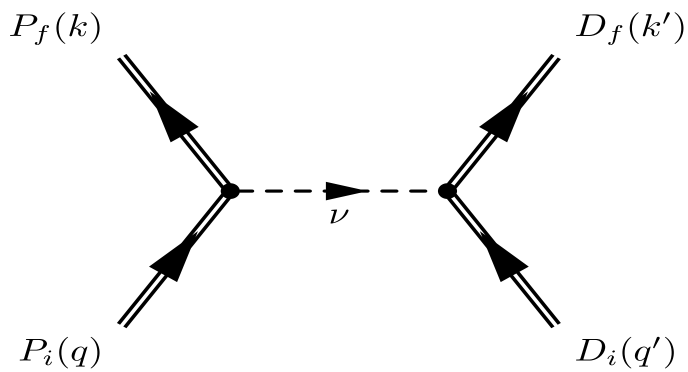
Transition amplitude:

$$\mathcal{A}_{\alpha\beta} \;=\; \sum_j U^*_{\alpha j} U_{\beta j} \mathcal{A}_j\,.$$

Use the Feynman rules in the configuration space. In lowest (2nd) order in weak interaction:

$$\mathcal{A}_j = \int d^4x_1 \int d^4x_2\, A^P_j(x_1) S_{Fj}(x_1 - x_2) A^D_j(x_2)\,.$$

How is it obtained?



$$|P_i\rangle = \int [dq]\, f_{Pi}(\vec{q}, \vec{Q})\, |P_i, \vec{q}\rangle\,, \qquad |P_f\rangle = \int [dk]\, f_{Pf}(\vec{k}, \vec{K})\, |P_f, \vec{k}\rangle\,,$$

$$|D_i\rangle = \int [dq']\, f_{Di}(\vec{q}', \vec{Q}')\, |D_i, \vec{q}'\rangle\,, \qquad |D_f\rangle = \int [dk']\, f_{Df}(\vec{k}', \vec{K}')\, |D_f, \vec{k}'\rangle\,.$$

The transition amplitude:

$$i\mathcal{A}_{\alpha\beta} = \langle P_f\, D_f | \hat{T} \exp\left[-i \int d^4x\, \mathcal{H}_I(x) \right] - \mathbb{1} | P_i\, D_i \rangle\,,$$

QFT approach – contd.

In the second order in weak interaction:

$$
\begin{aligned}
i\mathcal{A}_{\alpha\beta} \;\; &= \;\; \sum_j U^*_{\alpha j} U_{\beta j} \int [dq]\, f_{Pi}(\vec{q}, \vec{Q}) \int [dk]\, f^*_{Pf}(\vec{k}, \vec{K}) \\
&\times \int [dq']\, f_{Di}(\vec{q}\,', \vec{Q}') \int [dk']\, f^*_{Df}(\vec{k}', \vec{K}')\, i\mathcal{A}_j^{p.w.}(q, k; q', k')\,.
\end{aligned}
$$

Plane-wave amplitude:

$$
\begin{aligned}
i\mathcal{A}_j^{p.w.}(q, k; q', k') &= \int d^4x_1 \int d^4x_2\, \tilde{M}_D(q', k')\, e^{-i(q'-k')(x_2-x_D)} \\
&\times i \int \frac{d^4p}{(2\pi)^4} \frac{\not{p} + m_j}{p^2 - m_j^2 + i\epsilon}\, e^{-ip(x_2-x_1)}\, \tilde{M}_P(q, k)\, e^{-i(q-k)(x_1-x_P)}
\end{aligned}
$$

$\tilde{M}_{jP}$, $\tilde{M}_{jD}$ – production and detection amplitudes with neutrino spinors excluded. Full amplitudes:

$$
M_{jP}(q, k) \equiv \frac{\bar{u}_{jL}(p)}{\sqrt{2p_0}} \tilde{M}_P(q, k)\,, \qquad M_{jD}(q', k') \equiv \tilde{M}_D(q', k') \frac{u_{jL}(p)}{\sqrt{2p_0}}
$$

QFT approach – contd.

Neutrino prod. and det. regions: the overlap regions of the wave packets of participating external particles. 4-coordinates of the "central points" of these regions (points of the maximal overlap of external w. packets): x_P and x_D. It will be convenient to go to shifted 4-coordinates:

$$x_1' = x_1 - x_P\,, \qquad x_2' = x_2 - x_D\,.$$

Also define

$$T = t_D - t_P\,, \qquad \vec{L} = \vec{x}_D - \vec{x}_P\,.$$

A useful formula:

$$\not{p} + m_j = \sum_\sigma u_{j\sigma}(p)\bar{u}_{j\sigma}(p)\,.$$

For neutrinos only one chirality contributes $(\sigma = L$ for ν and $\sigma = R$ for $\bar{\nu})$ because of the chiral nature of weak interactions $\Rightarrow$ the sum over σ can be dropped; $u_{j\sigma}(p)$ and $\bar{u}_{j\sigma}(p)$ can then be merged with $\tilde{M}_{P,D}$ to produce M_{jP} and M_{jD}.

QFT approach – contd.

$$iA_{\alpha\beta} = i\sum_j U_{\alpha j}^* U_{\beta j} \int \frac{d^4p}{(2\pi)^4}\,\Phi_{jP}(p^0,\vec{p})\Phi_{jD}(p^0,\vec{p})\,\frac{2p_0\,e^{-ip^0T+i\vec{p}\vec{L}}}{p^2-m_j^2+i\epsilon}\,.$$

$$\Phi_{jP}(p^0,\vec{p}) = \int d^4x_1' e^{ipx_1'}\int[dq]\int[dk]\,f_{Pi}(\vec{q},\vec{Q})\,f_{Pf}^*(\vec{k},\vec{K})\,e^{-i(q-k)x_1'}M_{jP}(q,k)$$

$$\Phi_{jD}(p^0,\vec{p}) = \int d^4x_2' e^{-ipx_2'}\int[dq']\int[dk']\,f_{Di}(\vec{q}',\vec{Q}')f_{Df}^*(\vec{k}',\vec{K}')\,e^{-i(q'-k')x_2'}M_{jD}(q',k')$$

For $L \gg 1/p$ – fast oscillating factor in $iA_{\alpha\beta}$ $\Rightarrow$ main contribution to integral over p^0 from the pole at $p^0 = E_j(\vec{p}) - i\epsilon$ (on-shell neutrinos).

$$\Downarrow$$

$$\boxed{iA_{\alpha\beta} = \Theta(T)\sum_j U_{\alpha j}^* U_{\beta j}\int \frac{d^3p}{(2\pi)^3}\,\Phi_{jP}(E_j(\vec{p}),\vec{p})\Phi_{jD}(E_j(\vec{p}),\vec{p})\,e^{-iE_j(\vec{p})T+i\vec{p}\vec{L}}}$$

Comparing QM w.p. and QFT amplitudes

The transition amplitude:

$$\mathcal{A}_{\alpha\beta}(T,\vec{L}) \;=\; \sum_j U_{\alpha j}^* U_{\beta j}\, \mathcal{A}_j(T,\vec{L})$$

In QM WP approach we had:

$$\boxed{\mathcal{A}_j(T,\vec{L}) \;=\; \int \frac{d^3p}{(2\pi)^3}\, f_j^S(\vec{p})\, f_j^{D*}(\vec{p})\, e^{-iE_j(p)T+i\vec{p}\vec{L}}}$$

In QFT approach:

$$\boxed{i\mathcal{A}_j = \int \frac{d^3p}{(2\pi)^3}\, \Phi_{jP}(E_j(\vec{p}),\vec{p})\Phi_{jD}(E_j(\vec{p}),\vec{p})\, e^{-iE_j(\vec{p})T+i\vec{p}\vec{L}}}$$

The QM and QFT expressions have exactly the same form !

QFT approach – contd.

Comparing with $\mathcal{A}_{ab}(T, \vec{L})$ obtained in the QM w. packet approach: the two amplitudes essentially coincide if

$$f_{jP}(\vec{p}) = \Phi_{jP}(E_j(\vec{p}), \vec{p})\,, \qquad f_{jD}(\vec{p}) = \Phi^*_{jD}(E_j(\vec{p}), \vec{p})\,,$$

QFT approach – contd.

Comparing with $\mathcal{A}_{ab}(T,\vec{L})$ obtained in the QM w. packet approach: the two amplitudes essentially coincide if

$$f_{jP}(\vec{p}) = \Phi_{jP}(E_j(\vec{p}),\vec{p})\,, \qquad f_{jD}(\vec{p}) = \Phi^{*}_{jD}(E_j(\vec{p}),\vec{p})\,,$$

Easy to understand: $\Phi_{jP}(E_j(p),\vec{p})$ is the probability amplitude of ν production process in which ν_j is emitted with momentum $\vec{p}$

$\Rightarrow$ Φ_{jP} is momentum distribution amplitude of the produced neutrino, i.e. the momentum-state wave packet $f_{jP}(\vec{p})$. Similarly for neutrino detection.

N.B.: $f_{jP}(\vec{p})$ and $f_{jD}(\vec{p})$ are not “canonically” normalized.

Alternative approaches:

QFT approach – contd.

Comparing with $\mathcal{A}_{ab}(T,\vec{L})$ obtained in the QM w. packet approach: the two amplitudes essentially coincide if

$$f_{jP}(\vec{p}) = \Phi_{jP}(E_j(\vec{p}),\vec{p})\,, \qquad f_{jD}(\vec{p}) = \Phi^*_{jD}(E_j(\vec{p}),\vec{p})\,,$$

Easy to understand: $\Phi_{jP}(E_j(p),\vec{p})$ is the probability amplitude of ν production process in which ν_j is emitted with momentum $\vec{p}$

$\Rightarrow$ Φ_{jP} is momentum distribution amplitude of the produced neutrino, i.e. the momentum-state wave packet $f_{jP}(\vec{p})$. Similarly for neutrino detection.

N.B.: $f_{jP}(\vec{p})$ and $f_{jD}(\vec{p})$ are not “canonically” normalized.

Alternative approaches:

- $|P_f\nu_j\rangle \;=\; (S-\mathbb{1})|P_i\rangle\,, \qquad |\nu_j\rangle \;=\; \langle P_f|P_f\nu_j\rangle$

QFT approach – contd.

Comparing with $\mathcal{A}_{ab}(T,\vec{L})$ obtained in the QM w. packet approach: the two amplitudes essentially coincide if

$$f_{jP}(\vec{p}) = \Phi_{jP}(E_j(\vec{p}),\vec{p})\,, \qquad f_{jD}(\vec{p}) = \Phi^*_{jD}(E_j(\vec{p}),\vec{p})\,,$$

Easy to understand: $\Phi_{jP}(E_j(p),\vec{p})$ is the probability amplitude of ν production process in which ν_j is emitted with momentum $\vec{p}$
$\Rightarrow$ Φ_{jP} is momentum distribution amplitude of the produced neutrino, i.e. the momentum-state wave packet $f_{jP}(\vec{p})$. Similarly for neutrino detection.
N.B.: $f_{jP}(\vec{p})$ and $f_{jD}(\vec{p})$ are not "canonically" normalized.

Alternative approaches:

- $|P_f\nu_j\rangle \;=\; (S-\mathbb{1})|P_i\rangle\,, \qquad |\nu_j\rangle \;=\; \langle P_f|P_f\nu_j\rangle$
- In coord. space: ψ_{ν_j} = convolution of the ν source (prod. amplitude) and retarded propagator

QFT approach – contd.

Comparing with $\mathcal{A}_{ab}(T, \vec{L})$ obtained in the QM w. packet approach: the two amplitudes essentially coincide if

$$f_{jP}(\vec{p}) = \Phi_{jP}(E_j(\vec{p}), \vec{p})\,, \qquad f_{jD}(\vec{p}) = \Phi^*_{jD}(E_j(\vec{p}), \vec{p})\,,$$

Easy to understand: $\Phi_{jP}(E_j(p), \vec{p})$ is the probability amplitude of ν production process in which ν_j is emitted with momentum $\vec{p}$
$\Rightarrow$ Φ_{jP} is momentum distribution amplitude of the produced neutrino, i.e. the momentum-state wave packet $f_{jP}(\vec{p})$. Similarly for neutrino detection.
N.B.: $f_{jP}(\vec{p})$ and $f_{jD}(\vec{p})$ are not “canonically” normalized.

Alternative approaches:

- $|P_f \nu_j\rangle = (S - \mathbb{1})|P_i\rangle\,, \qquad |\nu_j\rangle = \langle P_f|P_f \nu_j\rangle$
- In coord. space: ψ_{ν_j} = convolution of the ν source (prod. amplitude) and retarded propagator

All three approaches give the same results.

General properties of ν w. packets in QFT

$$f_{jP}(\vec{p}) \simeq M_{jP}(Q,K) \int d^4x\, e^{iE_j(\vec{p})t - i\vec{p}\vec{x}} \int [dq] \int [dk] f_{Pi}(\vec{q},\vec{Q}) f^*_{Pf}(\vec{k},\vec{K}) e^{-i(q-k)x}$$

Integral over $\vec{x}$ gives $\sim \delta^{(3)}(\vec{q}-\vec{k}-\vec{p})$. Since $f_{Pi}(\vec{q},\vec{Q})$, $f_{Pf}(\vec{k},\vec{K})$ are sharply peaked at $\vec{Q}$ and $\vec{K}$ $\Rightarrow$ $f_{jP}(\vec{p})$ is sharply peaked at

$\vec{P} \equiv \vec{Q} - \vec{K}$. Width of the peak: $\sigma_{pP} \simeq \max\{\sigma_{P_i}, \sigma_{P_f}\}$

For external particles described by plane waves:

$$f_{jP}(\vec{p}) = \frac{M_{jP}(Q,K)}{\sqrt{2E_{Pi}V \cdot 2E_{Pf}V}} \delta^{(4)}(Q-K-p)$$

In general: $f_{jP}(\vec{p}) \Rightarrow M_{jP}(Q,K) \times$ ("smeared δ-functions") representing approx. conservation of mean energies and mean momenta.

Matching QM & QFT expressions for ν w. p.

Example – Gaussian wave packets for external particles. QFT gives

$$f_{jP}(\vec{p}) \;\propto\; [M_{jP}(Q,K)]/(\sigma_{eP}\sigma_{pP}^3)\,\exp\big[-g_P(E_j(\vec{p}),\,\vec{p})\big],$$

$$g_P(E_j(\vec{p}),\,\vec{p}) = \frac{(\vec{p}-\vec{P})^2}{4\sigma_{pP}^2} + \frac{[E_j(\vec{p}) - E_P - \vec{v}_P(\vec{p}-\vec{P})]^2}{4\sigma_{eP}^2}\,.$$

Here $\vec{P} \equiv \vec{Q} - \vec{K}\,, \quad E_P \equiv E_{Pi}(\vec{Q}) - E_{Pf}(\vec{K})\,,$

$$\sigma_{pP}^2 = \sigma_{pPi}^2 + \sigma_{pPf}^2\,, \qquad \sigma_{xP}\sigma_{pP} = \frac{1}{2}\,,$$

$$\vec{v}_P \equiv \sigma_{xP}^2\left(\frac{\vec{v}_{Pi}}{\sigma_{xPi}^2} + \frac{\vec{v}_{Pf}}{\sigma_{xPf}^2}\right), \qquad \Sigma_P \equiv \sigma_{xP}^2\left(\frac{\vec{v}_{Pi}^2}{\sigma_{xPi}^2} + \frac{\vec{v}_{Pf}^2}{\sigma_{xPf}^2}\right),$$

$$\sigma_{eP}^2 = \sigma_{pP}^2(\Sigma_P - \vec{v}_P^2) \equiv \sigma_{pP}^2\,\lambda_P\,, \qquad 0 \le \lambda_P \le 1\,.$$

For 2 ext. particles at production: $\sigma_{eP} = |\vec{v}_{Pi} - \vec{v}_{Pf}|/2\sqrt{\sigma_{xPi}^2 + \sigma_{xPf}^2}$ $\sim$ inverse overlap time

Matching QM & QFT expressions for ν w. p.

Compare with Gaussian wave packet in QM approach:

$$f_{jP}(\vec{p},\vec{P}) = \left(\frac{2\pi}{\sigma_{pP}^2}\right)^{3/4} \exp\left[-\frac{(\vec{p}-\vec{P})^2}{4\sigma_{pP}^2}\right]$$

To match the QM and QFT expression: expand $E_j(\vec{p})$ around $\vec{p}=\vec{P}$ and subst. into $g_P(E_j(\vec{p}),\vec{p})$:

♢ $$\boxed{g_P(E_j(\vec{p}),\vec{p}) = (p-P)^k\,\alpha^{kl}\,(p-P)^l - \beta^k(p-P)^k + \gamma_j}$$

$$\alpha^{kl} = \frac{1}{4\sigma_{eP}^2}\left[\lambda_P\,\delta^{kl} + (v_j-v_P)^k\,(v_j-v_P)^l + \frac{E_j-E_P}{E_j}(\delta^{kl}-v_j^k v_j^l)\right],$$

$$\beta^k = -\frac{1}{2\sigma_{eP}^2}(E_j-E_P)(v_j-v_P)^k\,, \qquad \gamma_j = \frac{(E_j-E_P)^2}{4\sigma_{eP}^2}\,.$$

Try to represent $g_P(E_j(\vec{p}),\vec{p})$ in the form

♢ $$\boxed{g_P(E_j(\vec{p}),\vec{p}) = (p-P_{\rm eff})^k\,\alpha^{kl}\,(p-P_{\rm eff})^l + \tilde{\gamma}_j}\,, \qquad \vec{P}_{\rm eff} \equiv \vec{P}+\vec{\delta}$$

Matching QM & QFT expressions for ν w. p.

$$\delta^k = -\frac{(E_j - E_P)(v_j - v_P)^k}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}\,, \qquad \tilde{\gamma}_j = \frac{(E_j - E_P)^2}{4\sigma_{eP}^2}\,\frac{\lambda_P}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}\,.$$

Diagonalization of α^{kl} gives $(OZ||(\vec{v}_j - \vec{v}_P))$:

$$(\sigma^x_{pP\,\mathrm{eff}})^2 = (\sigma^y_{pP\,\mathrm{eff}})^2 = \sigma_{pP}^2\,, \qquad \frac{1}{(\sigma^z_{pP\,\mathrm{eff}})^2} = \frac{1}{\sigma_{pP}^2} + \frac{(\vec{v}_j - \vec{v}_P)^2}{\sigma_{eP}^2}\,,$$

$\Rightarrow$ QM neutrino wave packets can match those obtained QFT if

Matching QM & QFT expressions for ν w. p.

$$\delta^k = -\frac{(E_j - E_P)(v_j - v_P)^k}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}\,, \qquad \tilde{\gamma}_j = \frac{(E_j - E_P)^2}{4\sigma_{eP}^2}\,\frac{\lambda_P}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}\,.$$

Diagonalization of α^{kl} gives ($OZ||(\vec{v}_j - \vec{v}_P)$):

$$(\sigma_{pP\,\mathrm{eff}}^x)^2 = (\sigma_{pP\,\mathrm{eff}}^y)^2 = \sigma_{pP}^2\,, \qquad \frac{1}{(\sigma_{pP\,\mathrm{eff}}^z)^2} = \frac{1}{\sigma_{pP}^2} + \frac{(\vec{v}_j - \vec{v}_P)^2}{\sigma_{eP}^2}\,,$$

$\Rightarrow$ QM neutrino wave packets can match those obtained QFT if

- Momentum uncertainties of the neutrino mass eigenstates are replaced (anisotropic) effective ones: $-(\vec{p} - \vec{P})^2/(4\sigma_{pP}^2) \;\to$

$-[(p^x - P_{\mathrm{eff}}^x)^2/4(\sigma_{pP}^x)^2 + (p^y - P_{\mathrm{eff}}^y)^2/4(\sigma_{pP}^y)^2 + (p^z - P_{\mathrm{eff}}^z)^2/4(\sigma_{pP}^z)^2]$.

Matching QM & QFT expressions for ν w. p.

$$\delta^k = -\frac{(E_j - E_P)(v_j - v_P)^k}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}\,, \qquad \tilde{\gamma}_j = \frac{(E_j - E_P)^2}{4\sigma_{eP}^2}\,\frac{\lambda_P}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}\,.$$

Diagonalization of α^{kl} gives $(OZ||(\vec{v}_j - \vec{v}_P))$:

$$(\sigma_{pP\,\mathrm{eff}}^x)^2 = (\sigma_{pP\,\mathrm{eff}}^y)^2 = \sigma_{pP}^2\,, \qquad \frac{1}{(\sigma_{pP\,\mathrm{eff}}^z)^2} = \frac{1}{\sigma_{pP}^2} + \frac{(\vec{v}_j - \vec{v}_P)^2}{\sigma_{eP}^2}\,,$$

$\Rightarrow$ QM neutrino wave packets can match those obtained QFT if

- Momentum uncertainties of the neutrino mass eigenstates are replaced (anisotropic) effective ones: $-(\vec{p} - \vec{P})^2/(4\sigma_{pP}^2) \;\rightarrow$

  $-[(p^x - P_{\mathrm{eff}}^x)^2/4(\sigma_{pP}^x)^2 + (p^y - P_{\mathrm{eff}}^y)^2/4(\sigma_{pP}^y)^2 + (p^z - P_{\mathrm{eff}}^z)^2/4(\sigma_{pP}^z)^2]$.
- The mean momentum $\vec{P}$ is shifted according to $\vec{P} \rightarrow \vec{P}_{\mathrm{eff}} = \vec{P} + \vec{\delta}$.

Matching QM & QFT expressions for ν w. p.

$$\delta^k = -\frac{(E_j - E_P)(v_j - v_P)^k}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}\,, \qquad \tilde{\gamma}_j = \frac{(E_j - E_P)^2}{4\sigma_{eP}^2}\frac{\lambda_P}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}\,.$$

Diagonalization of α^{kl} gives $(OZ||(\vec{v}_j - \vec{v}_P))$:

$$(\sigma_{pP\,\mathrm{eff}}^x)^2 = (\sigma_{pP\,\mathrm{eff}}^y)^2 = \sigma_{pP}^2\,, \qquad \frac{1}{(\sigma_{pP\,\mathrm{eff}}^z)^2} = \frac{1}{\sigma_{pP}^2} + \frac{(\vec{v}_j - \vec{v}_P)^2}{\sigma_{eP}^2}\,,$$

$\Rightarrow$ QM neutrino wave packets can match those obtained QFT if

- Momentum uncertainties of the neutrino mass eigenstates are replaced (anisotropic) effective ones: $-(\vec{p} - \vec{P})^2/(4\sigma_{pP}^2) \;\to$

  $-[(p^x - P_{\mathrm{eff}}^x)^2/4(\sigma_{pP}^x)^2 + (p^y - P_{\mathrm{eff}}^y)^2/4(\sigma_{pP}^y)^2 + (p^z - P_{\mathrm{eff}}^z)^2/4(\sigma_{pP}^z)^2]$.
- The mean momentum $\vec{P}$ is shifted according to $\vec{P} \to \vec{P}_{\mathrm{eff}} = \vec{P} + \vec{\delta}$.
- The wave packet of each neutrino mass eigenstate gets an extra factor $N_j = \exp[-\tilde{\gamma}_j]$.

Matching QM & QFT expressions for ν w. p.

If $\qquad |E_i - E_j| \ll \sigma_{eP} \quad \Rightarrow$

factors N_j are the same for all ν mass eigenstates, can be included in common normalization factor. In the opposite case – coherence of different neutrino mass eigenstates is lost.

$\sigma_{eP} \le \sigma_{pP} \quad \Rightarrow$ except for $\vec{v}_j \approx \vec{v}_P$ momentum uncertainty along $(\vec{v}_j - \vec{v}_P)$ is dominated by σ_{eP}.

In the stationary neutrino source limit $(\sigma_{eP}\,, \vec{v}_P \to 0)$, effective longitudinal mom. uncertainty $\sigma^z_{pP\,\mathrm{eff}} = 0$ even though the true mom. uncertainty $\sigma_{pP} \neq 0$.

$$\Downarrow$$

Coherence length $\quad l_{\mathrm{coh}} \; \to \; \infty$

Oscillation probability in QFT

What is calculated in QFT is the probability of the overall production-propagation-detection process. How to extract from it the oscillation probability $P_{\alpha\beta}(L)$?

1. Recall the operational definition of $P_{\alpha\beta}(L)$. Detection rate for ν_β:

$$\Gamma_\beta^{\rm det} = \int dE\, j_\beta(E)\sigma_\beta(E)\,,$$

If a source at a distance L from the detector emits ν_α with the energy spectrum $d\Gamma_\alpha^{\rm prod}(E)/dE$:

$$j_\beta(E) = \frac{1}{4\pi L^2}\frac{d\Gamma_\alpha^{\rm prod}(E)}{dE} P_{\alpha\beta}(L,E)\,,$$

$\Rightarrow$ substitute into $\Gamma_\beta^{\rm det}$:

Oscillation probability in QFT

$$\Gamma^{\rm tot}_{\alpha\beta} \equiv \int dE\, \frac{d\Gamma^{\rm tot}_{\alpha\beta}(E)}{dE} = \frac{1}{4\pi L^2}\int dE\, \frac{d\Gamma^{\rm prod}_{\alpha}(E)}{dE}\, P_{\alpha\beta}(L,E)\,\sigma_\beta(E)$$

$$P_{\alpha\beta}(L,E) = \frac{d\Gamma^{\rm tot}_{\alpha\beta}(E)/dE}{\frac{1}{4\pi L^2}\,[d\Gamma^{\rm prod}_{\alpha}(E)/dE]\,\sigma_\beta(E)}\,.$$

An important ingredient: the assumption that the overall rate factorizes into the production rate, propagation (oscillation) probability and detection cross section.

If this does not hold, oscillation probability is undefined $\Rightarrow$

Need to deal instead with the overall rate of neutrino production, propagation and detection.

Oscillation probability in QFT

Try to cast $P_{\alpha\beta}^{\rm tot}$ in the same form (check if the factorization condition holds !)

$$\boxed{i\mathcal{A}_{\alpha\beta} = i\sum_j U_{\alpha j}^* U_{\beta j}\int \frac{d^4p}{(2\pi)^4}\,\Phi_{jP}(p^0,\vec{p})\Phi_{jD}(p^0,\vec{p})\,\frac{2p_0\,e^{-ip^0T+i\vec{p}\vec{L}}}{p^2-m_j^2+i\epsilon}}$$

Integrate first over $\vec{p}$, then over $p^0\equiv E$. Make use of Grimus-Stockinger theorem: for a large L ($L\gg p/\sigma_p^2$), $A>0$ and a sufficiently smooth $\psi(\vec{p})$,

$$\int d^3p\frac{\psi(\vec{p})\,e^{i\vec{p}\vec{L}}}{A-\vec{p}^2+i\epsilon} = -\frac{2\pi^2}{L}\psi(\sqrt{A}\tfrac{\vec{L}}{L})e^{i\sqrt{A}L}+\mathcal{O}(L^{-\frac{3}{2}}) \quad \Rightarrow$$

$$i\mathcal{A}_{\alpha\beta}(T,\vec{L}) = \frac{-i}{8\pi^2 L}\sum_j U_{\alpha j}^* U_{\beta j}\int dE\,\Phi_P(E,p_j\vec{l})\Phi_D(E,p_j\vec{l})\;2E\,e^{-iE\,T+ip_jL}$$

where

$$p_j\equiv\sqrt{E^2-m_j^2}\,,\qquad \vec{l}\equiv\frac{\vec{L}}{L}\,,$$

Oscillation probability in QFT

Introduce

$$\tilde{P}^{\rm tot}_{\alpha\beta}(\vec{L}) = \int dT\, P_{\alpha\beta}(T,\vec{L}) = \frac{1}{8\pi^2}\frac{1}{4\pi L^2}\sum_{j,k} U^*_{\alpha j}U_{\beta j}U_{\alpha k}U^*_{\beta k}$$

$$\times \int dE\, \Phi_P(E, p_j\vec{l})\Phi_D(E,p_j\vec{l})\, \Phi^*_P(E,p_k\vec{l})\Phi^*_D(E,p_k\vec{l})\ (2E)^2\, e^{i(p_j-p_k)L}$$

Neutrino production probability:

$$P^{\rm prod}_{\alpha} = \sum_j |U_{\alpha j}|^2 \int \frac{d^3p_j}{(2\pi)^3}\,\left|\Phi_P(E,p_j)\right|^2 = \sum_j |U_{\alpha j}|^2 \frac{1}{8\pi^2}\int dE\,\left|\Phi_P(E,p_j)\right|^2 4Ep_j$$

Detection probability:

$$P^{\rm det}_{\beta}(E) = \sum_k |U_{\beta k}|^2 |\Phi_D(E,p_k)|^2 \frac{1}{V}\,,$$

Oscillation probability in QFT

Let the number of particles P_i entering the production region during time interval T_0 be N_P and number of D_i entering the detection region be N_D. Probability of neutrino emission during the finite interval of time t:

$$\mathcal{P}_\alpha^{\rm prod}(t) = N_P \int_0^t \frac{dt_P}{T_0}\, P_\alpha^{\rm prod} = N_P\, P_\alpha^{\rm prod}\frac{t}{T_0}\,, \quad \text{rate:} \quad \Gamma_\alpha^{\rm prod} = N_P\, P_\alpha^{\rm prod}\,\frac{1}{T_0}$$

Detection cross section:

$$\sigma_\beta(E) = \frac{N_D}{T_0}\sum_k |U_{\beta k}|^2 |\Phi_{kD}(E)|^2 \frac{E}{p_k}$$

Probability of the overall production-propagation-detection process:

$$\mathcal{P}_{\alpha\beta}^{\rm tot}(t,L) = \frac{N_P N_D}{T_0^2}\int_0^t dt_D \int_0^t dt_P\, P_{\alpha\beta}^{\rm tot}(T,L) \quad \Rightarrow$$

Oscillation probability in QFT

New integration variables $\tilde{T} \equiv (t_P + t_D)/2$ and $T = t_D - t_P \quad \Rightarrow$

$$\mathcal{P}^{\rm tot}_{\alpha\beta}(t, L) = \frac{N_P N_D}{T_0^2}\Big[\int_0^t dT\, P^{\rm tot}_{\alpha\beta}(T, L)(t-T) + \int_{-t}^0 dT\, P^{\rm tot}_{\alpha\beta}(T, L)(t+T)\Big]$$

$$= \frac{N_P N_D}{T_0^2}\Big[t\int_{-t}^t dT\, P^{\rm tot}_{\alpha\beta}(T, L) - \int_0^t dT\, T\, P^{\rm tot}_{\alpha\beta}(T, L) + \int_{-t}^0 dT\, T\, P^{\rm tot}_{\alpha\beta}(T, L)\Big]$$

$$\equiv \frac{N_P N_D}{T_0^2}\Big[t I_1(t) - I_2(t) + I_3(t)\Big].$$

For large t (much larger than the time scales of the neutrino production and detection processes) $I_1 = \tilde{P}^{\rm tot}_{\alpha\beta}(L)$ whereas $I_2 = I_3 = 0 \quad \Rightarrow$

$$\mathcal{P}^{\rm tot}_{\alpha\beta}(t, L) = \frac{N_P N_D}{T_0^2}\, t\, \tilde{P}^{\rm tot}_{\alpha\beta}(L)\,, \qquad \Gamma^{\rm tot}_{\alpha\beta}(L) = \frac{N_P N_D}{T_0^2}\, \tilde{P}^{\rm tot}_{\alpha\beta}$$

Oscillation probability in QFT

$$\text{“}P_{\alpha\beta}(L,E)\text{”} = \frac{\sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \Phi_P(E,p_j)\Phi_D(E,p_j)\Phi_P^*(E,p_k)\Phi_D^*(E,p_k)e^{i(p_j-p_k)L}}{\sum_j |U_{\alpha j}|^2\,|\Phi_P(E,p_j)|^2\,p_j\;\sum_k |U_{\beta k}|^2\,|\Phi_D(E,p_k)|^2 p_k^{-1}}$$

For $|p_j - p_k| \ll p_j, p_k$ (ultra-relativistic or quasi-degenerate in mass ν's) and

$$\boxed{|p_j - p_k| \ll \sigma_{pP}, \sigma_{pD}}$$

one can replace

$$p_j \to p\,, \qquad \Phi_P(E,p_j) \to \Phi_P(E,p) \qquad (p\;-\;\text{average momentum})$$

$\Rightarrow$ in the denominator of “$P_{\alpha\beta}(L,E)$”:

$$\sum_j |U_{\alpha j}|^2\,|\Phi_P(E,p_j)|^2\,p_j \to |\Phi_P(E,p)|^2\,p\sum_j |U_{\alpha j}|^2 = |\Phi_P(E,p)|^2\,p\,,$$

$$\sum_k |U_{\beta j}|^2\,|\Phi_D(E,p_k)|^2\,p_k^{-1} \to |\Phi_D(E,p)|^2\,p^{-1}\sum_k |U_{\beta k}|^2 = |\Phi_D(E,p)|^2\,p^{-1}\,,$$

Oscillation probability in QFT

In the numerator of "$P_{\alpha\beta}(L,E)$" Φ_P, Φ_D can be pulled out of the sums and canceled with those in the denominator. $\Rightarrow$ stand. osc. probability:

$$\boxed{P_{\alpha\beta}(L,E) = \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \, e^{-i\frac{\Delta m_{jk}^2}{2p}L}}$$

Automatically satisfies unitarity, i.e. is properly normalized.

For $|p_j - p_k| \gg \sigma_p$ $(\Leftrightarrow \Delta m_{jk}^2/(2p) \gg \sigma_p)$ – interf. terms strongly suppressed $\Rightarrow$ Decoherence

Oscillation probability in QFT

The condition for the existence of well-defined oscillation probabilities is that neutrinos are either ultra-relativistic or nearly degenerate in mass and, in addition, the coherence prod./detection conditions are satisfied:

$$\boxed{|p_j - p_k| \ll \sigma_{pP}, \sigma_{pD}}$$

The QFT-based consideration clarifies the QM wave packet normalization prescription. QM and QFT approaches can be matched if the QM quantities f_{jP} and f_{jD} are identified with the QFT functions $\Phi_{jP}(E_j, \vec{p})$ and $\Phi^*_{jD}(E_j, \vec{p})$, respectively. But: the latter bear information not only on the properties of the emitted and absorbed neutrinos, but also on the production and detection processes. The QM normalization procedure is equivalent, in the limit $|p_j - p_k| \ll \sigma_{pP}, \sigma_{pD}$, to the division of the overall rate of the process by the production rate and detection cross section, as in QFT approach.