General Relativity and Gravitational Waves

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These are lecture notes for the two lectures on *General Relativity and Gravitational Waves* given at the *Cargèse School on Gravitational Waves*, on Monday May, 23$^{\text{rd}}$ 2011. They are really simple notes to keep track of the equations and the overall structure of the lecture, in particular they do not contain proofs of the results, nor detailed explanations. They are supposed to be an introduction to the more detailed lectures by Pr. Bernard Schutz (*Astrophysics of Sources of Gravitational waves*) and Pr. Alesandra Buonanno (*Models of Gravitational Waves*).

Although these introductory lectures should be quite general, many of the results presented here are aimed toward an application to astrophysical systems. In particular, no cosmological solution is presented. In both lectures Greek indices $(\alpha, \beta, \dots \mu, \nu, \dots)$ are spacetime indices ranging from 0 to 3, whereas Latin ones $(i, j, \dots)$ range only from 1 to 3 for spatial indices (in particular in Sec. 1.4). In addition, Einstein summation convention over repeated indices shall be used:

$$A_{\alpha\beta}\, \xi^\alpha = \sum_{\alpha=0}^{4} A_{\alpha\beta}\, \xi^\alpha.$$

There are many books about the theory of general relativity. Only a few of them are cited here for the interested reader:

- L.N. Landau & E.M. Lifshitz *The classical theory of fields*, Pergamon Press
- C.W. Misner, K.S. Thorne & J.A. Wheeler *Gravitation*, Freeman
- R.M. Wald *General Relativity*, University of Chicago Press
- S. Weinberg *Gravitation and Cosmology*, Wiley
- S. Caroll *Spacetime and Geometry: An introduction to General Relativity*, Addison-Wesley
- M. Alcubierre *Introduction to 3+1 Numerical Relativity*, Oxford Science Publication
- E. Gourgoulhon *3+1 Formalism and Bases of Numerical Relativity*, `arXiv:gr-qc/0703035`

For those who can understand French, the Master course of General Relativity by E. Gourgoulhon at http://luth.obspm.fr/ luthier/gourgoulhon/fr/master/relatM2.pdf.

Chapter 1

Theoretical Foundations of General Relativity

1.1 Introduction

1.1.1 Newton's law

Among the four fundamental interactions of today's standard model in physics, gravitation was the first to be accurately described and modeled. Newton's law of universal gravitation (first published in 1687) states that two point-like massive bodies attract each other with a force $\vec{F}$ which amplitude is

$$F = \frac{G m_1 m_2}{r_{12}^2}, \tag{1.1}$$

where G is the gravitational constant, m_1, m_2 the masses of the two objects and r_{12} their relative distance.

Within this Newtonian model, gravitational interaction is transmitted instantaneously over all space. This was already of some concern to Isaac Newton, but it clearly became an issue with the development of the theory of special relativity (see Sec. 1.1.2 below). From the experimental side , Newton's law (1.1) is valid up to high accuracy until the masses are moving at relativistic speeds, or one is considering the gravitational field of *compact* objects (see Sec. 2.1.3 for a definition).

1.1.2 Special relativity

At the end of the 19$^{\text{th}}$ century, Abraham Michelson designed an experiment in order to detect ether[1]-induced effects, using what is now called a *Michelson interferometer* (see lecture by Pr. Jean-Yves Vinet) to measure the velocity of light coming from a source at two directions of the interferometer with respect to the motion of the Earth around the

[1] *ether* was a concept introduced by Maxwell as the medium on which the electromagnetic waves were propagating

Sun. The result of his experiment, and later with Edward Morley, was completely negative giving the same velocity of light at any direction. This was opening a major problem that could only be solved with the works leading to the theory of special relativity, as formulated by Albert Einstein in 1905.

This theory mixes notions of space and time, and relies on two postulates:

1. In vacuum, light propagates at the constant velocity c, independently of the movements of the source or of the observer;

2. All laws of physics have the same form in all inertial frames.

Without entering into this theory, it is important here to introduce the notion of *interval* between two events $\mathcal{P}_1$ and $\mathcal{P}_2$. Let us take a coordinate system linked with an inertial frame and each event shall be described by his 4 coordinates: $\mathcal{P}_1 = (ct_1, x_1, y_1, z_1)$ and $\mathcal{P}_2 = (ct_2, x_2, y_2, z_2)$, then the interval between both events is

$$s^2 = -c^2 (t_2 - t_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 . \tag{1.2}$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are infinitesimally close, $x_2^\alpha = x_1^\alpha + dx^\alpha$ then the infinitesimal interval is

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{1.3}$$

Any of these intervals is invariant under the action of Lorentz transforms, which ensures that light velocity is indeed the same in any inertial frame. From this property, it is possible to define the *lightcone* $\mathcal{C}_\mathcal{P}$ from an event $\mathcal{P}$ to be all the events which are at zero interval from $\mathcal{P}$, *i.e.* which can be reached by a light ray emitted at $\mathcal{P}$ (future lightcone), or which can reach $\mathcal{P}$ by a light ray emitted at them (past lightcone). A zero interval is called *null*, a positive one is *spacelike* and corresponds to events which are connected to $\mathcal{P}$ with velocities greater than c; a negative one is called *timelike* and corresponds to events which are connected to $\mathcal{P}$ with velocities smaller than c. This is called the (local) causal structure around $\mathcal{P}$.

1.1.3 Relativistic gravity?

Special relativity is a relevant framework to describe electromagnetic interactions, and also strong and weak interactions. Unfortunately, as far as gravitation is concerned, the situation is more complicated. There have been, of course, several attempts to get a (special) relativistic formulation of gravitation. If one writes that the force in Eq. (1.1) is the gradient of a potential Φ, then a common form of Newton's law is

$$\Delta\Phi = 4\pi G\, \rho, \tag{1.4}$$

with ρ the mass density. A straightforward relativistic extension of the Poisson equation (1.4) is a wave equation of the form

$$\Box\Phi = -\frac{4\pi G}{c^2} T, \tag{1.5}$$

where T is the trace of the stress-energy tensor describing the matter content (see Sec. 1.3.4). This scalar theory is relativistic and gives the right Newtonian limit (1.4) when $c \to +\infty$. However, this theory disagrees with observations such as the Mercury's perihelion precession (see Sec. 2.2.2), where it predicts a wrong sign for the effect. Furthermore, it does not predict any deviation of the light rays (see below), contrary to what has been observed by many experiments since 1919. More elaborated theories, in which the gravitational potential would be a vector or a tensor have severe problems too: in the vector case, the theory is unstable, and in the tensor case matter does not feel the gravitation it is generating!

It is therefore necessary to seek another model, and it is interesting to note that gravitation possesses the property of universality of free fall: all bodies are falling the same way, if not submitted to any other force. This is linked to the observed fact that the *inertial mass* of a body appearing in Newton's second law of dynamics is equal to its *gravitational mass* (or gravitational charge), independently of its composition. With a different formulation: a static and uniform gravitational field is equivalent to an accelerated frame. This has been elaborated by Einstein in his famous thought experiment: an observer freely falling in a lift cannot determine whether there is a gravitational field outside the lift. Nowadays, there are three equivalence principles that are used:

- **The weak equivalence principle**: given the same initial position and velocity, all point-like massive particles fall along the same trajectories.
- **The Einstein equivalence principle**: in a locally inertial frame, all non-gravitational laws of physics are given by their special-relativistic form.
- **The strong equivalence principle**: It is always possible to suppress the effects of an exterior gravitational field by choosing a locally inertial frame in which all laws of physics, including gravity, take the same form as in the absence of this exterior gravitational field.

It can be considered that the weak and the Einstein equivalence principles are equivalent, whereas the strong one only implies the two others. It also indicates that a relativistic theory verifying the Einstein equivalence principle should be non-linear.

With the relativistic notion that energy and mass are related, and the equivalence principle, a consequence is that time and space references may vary from one point of spacetime to another, in the presence of gravitational field. Such properties have indeed been observed, as it shall be detailed in Sec. 2.2.2. Let us consider two observers at rest with respect to each other: the first observer on Earth at some altitude z_0 is sending light signals with period T_0 to the second one, who is at the altitude $z_1 > z_0$. The second one receives signals with a period $T_1 > T_0$, meaning that clock signals received from a different gravitational potential are deformed. Furthermore, one may also expect, and it is observed, that light rays may be deflected in the vicinity of gravitating bodies, as the Sun or galaxies (gravitational lensing). The full structure of the special-relativistic spacetime (Minkowski spacetime) is determined by the lightcones (Sec. 1.1.2), which depend on the way light rays are propagating. Therefore, "deformed" lightcones in space and time let us think that gravitation can change space and time references: spacetime can thus appear as curved. Moreover, the notion of "straight line" comes from light rays and it therefore becomes meaningless if gravitation is present. The mathematical object best suited for such model is that of a manifold.

1.2 Manifold, metric and geodesics

1.2.1 Some definitions

A four-dimensional *manifold* $\mathcal{M}$ is a set of points that can be locally compared to $\mathbb{R}^4$ in the sense that one can assign four *coordinates* to every point of $\mathcal{M}$ and that these coordinates form a subset of $\mathbb{R}^4$, called *a chart*. A given manifold can need several charts to describe it and the coordinate choice is not in general unique: coordinate systems are arbitrary. Formally, for every point $P \in \mathcal{M}$, there exist a couple $(\mathcal{U}, \Psi)$, where $\mathcal{U}$ is an open subset of $\mathcal{M}$ and Ψ a map:

$$\begin{array}{rcl} \Psi: \ \mathcal{U} \subset \mathcal{M} & \rightarrow & \mathbb{R}^4 \\ P & \mapsto & (x^0, x^1, x^2, x^3)\,. \end{array} \tag{1.6}$$

The set of all $(\mathcal{U}_i, \Psi_i)$, where the $\{\mathcal{U}_i\}$'s cover all the manifold, is called an *atlas*. $\mathcal{M}$ is then called a *differentiable manifold* (smooth manifold) if, for every non-empty intersection $\mathcal{U}_i \cap \mathcal{U}_j$ the function $\Psi_i \circ \Psi_j^{-1}$ is differentiable (smooth).

Some common examples of **two**-dimensional manifolds include the cylinder and the sphere, for which at least two charts are always necessary. Note that a manifold does not need any higher-dimensional space to be embedded into: a 2-sphere can be looked at as a two-surface, forgetting about the $\mathbb{R}^3$ structure.

1.2.2 Vectors, forms and tensors

The notions of physical fields requires the generalization of scalars, vectors, . . . to the case of a manifold. The central idea here is the possibility to change the map, or coordinate system, on the manifold. Doing so, one would like to have the same form for the physical laws, in all possible coordinate systems. This is the notion of *covariance*, that generalizes the second principle of special relativity given in Sec. 1.1.2. Physical laws should therefore be expressed in terms of objects that transform in a well-defined manner, when changing from one coordinate system $\{x^\mu\}$ to another $\{x'^\mu\}$.

First, a *scalar* field is just a real-valued function $S(x^\mu)$ depending on the point on the manifold, that does not change under the change of coordinates

$$S'(x') = S(x). \tag{1.7}$$

Contrary to the affine space of special relativity, there cannot be any identification between a couple of points in the manifold and a vector. At every point P is defined a tangent space $\mathcal{T}_P$ in which vectors can be defined. The definition of a vector field on a manifold $\mathcal{M}$ can be given in two ways. First one can use the fact that the choice of coordinates on a manifold is arbitrary, the vector field $V^\mu(x^\mu)$ is then the field of elements of the vector space $\mathbb{R}^4$, which transform under the above mentioned change of coordinates as:

$$V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x). \tag{1.8}$$

Such a field is said to have one contravariant index, or to be a $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ tensor.

The second way of defining a vector field on a manifold is by using a curve $x^\mu = X^\mu(\lambda)$, with λ the parameter of the curve. A vector at a given point P on the curve is then the operator that assigns to every scalar field $f : \mathcal{M} \to \mathbb{R}$, its derivative along the curve:

$$\vec{V}(f) = \frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu} \frac{dX^\mu}{d\lambda}. \tag{1.9}$$

This can be seen as a directional derivative, and the vector is then given by this direction in the tangent space $\mathcal{T}_P$. Special tangent vectors are given by constant coordinate curves, *e.g.*

$$\begin{cases} x^0 &= \lambda \\ x^1 &= \text{constant} \\ x^2 &= \text{constant} \\ x^3 &= \text{constant} \end{cases}$$

for which

$$\vec{\partial}_0(f) = \frac{\partial f}{\partial x^0}. \tag{1.10}$$

Thus the four vectors $\left(\vec{\partial}_0, \vec{\partial}_1, \vec{\partial}_2, \vec{\partial}_3\right)$ form the natural base associated to the coordinates for every tangent space, and any vector field is thus defined through its components in this base:

$$\vec{V} = V^\mu \, \vec{\partial}_\mu = V^\mu \, \frac{\vec{\partial}}{\partial x^\mu}. \tag{1.11}$$

A *1-form* is a linear operator assigning to a vector a number. They are defined in dual space to $\mathcal{T}_P$, written $\mathcal{T}_P^*$. A form W_μ is also called a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor and is said to possess one covariant index. Under coordinate changes on the manifold $\mathcal{M}$, a 1-form transforms as:

$$W'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} W_\nu(x). \tag{1.12}$$

With these two definitions, it is possible to describe the most general tensor as a "tensor" product of vectors and forms.. A *p-times contravariant and q-times covariant*, or $\begin{pmatrix} p \\ q \end{pmatrix}$ tensor at a point P is written

$$T^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q}$$

and is a function from $\mathcal{T}_P^* \times \cdots \times \mathcal{T}_P^*(p \text{ times}) \times \mathcal{T}_P \times \cdots \times \mathcal{T}_P(q \text{ times})$ to $\mathbb{R}$, which is linear with respect to each argument, every contravariant index represents a vector-type behavior and every covariant one a form-like behavior. The tensor $T^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}$ is said to be of order (or rank) $p+q$. This most general tensor transforms under a change of coordinates in the following way:

$$T'^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}(x') = \frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\alpha_p}}{\partial x^{\mu_p}} \frac{\partial x^{\nu_1}}{\partial x'^{\beta_1}} \cdots \frac{\partial x^{\nu_q}}{\partial x'^{\beta_q}} T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}(x) \tag{1.13}$$

1.2.3 Metric

An important notion in vector spaces is the scalar product of two vectors. In special relativity, the scalar product includes the time coordinate to read

$$\vec{U} \cdot \vec{V} = -U^0 V^0 + U^1 V^1 + U^2 V^2 + U^3 V^3 = \eta_{\mu\nu} U^\mu V^\nu, \tag{1.14}$$

which defines $\eta_{\mu\nu}$. This symmetric 2-form is called the Minkowski *metric*, it is a fundamental object in special relativity and its generalization to the manifold case is even more important. At every point $P \in \mathcal{M}$, one defines a symmetric 2-form $g_{\mu\nu}$ acting on any couple of vectors of $\mathcal{T}_P$, and which is non-degenerate: if $\forall V^\nu \in \mathcal{T}_P$, $g_{\mu\nu} U^\mu V^\nu = 0$ then $U^\mu = 0$.

One can determine a base of $\mathcal{T}_P$ such that $g_{\mu\nu} = \eta_{\mu\nu}$ and the metric is said to have $(-,+,+,+)$ signature. $g_{\mu\nu}$ is said to be a *metric tensor* on $\mathcal{M}$ and $(\mathcal{M}, g_{\mu\nu})$ is called the spacetime. Returning now to the definition of an infinitesimal interval (1.3), one can write it in a general coordinate system on a manifold:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1.15}$$

which is the common way of defining a metric for a given spacetime. In order to measure the distance between two points P and P' on a spacetime which are not infinitesimally close, one must specify a curve joining both points and then integrate the element $\sqrt{\pm ds^2}$ along this curve. The result depend on the chosen curve, but not on the coordinate system. Similarly, the metric is used to compute angles between curves (or vectors) on the manifold.

As $g_{\mu\nu}$ is non-degenerate, one can define its inverse $g^{\mu\nu}$ such that

$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu. \tag{1.16}$$

The metric and its inverse are often used to "raise" and "lower" indices on tensors: through the definition of the scalar product and Eq. (1.16), they define a one-to-one relation (and its inverse) between vectors and forms:

$$U_\mu = g_{\mu\nu} U^\nu, \qquad W^\mu = g^{\mu\nu} W_\nu, \tag{1.17}$$

they are also used for "double contraction", to obtain the trace (a scalar) of rank 2 tensors:

$$g_{\mu\nu} T^{\mu\nu} = T. \tag{1.18}$$

With the metric, it is possible to define types for the vectors, as for intervals with the lightcone in Sec. 1.1.2. The norm squared of a vector U^μ is defined as $g_{\mu\nu}U^\mu U^\nu = U^\mu U_\mu$ and

- if $U^\mu U_\mu > 0$, the vector is said to be *spacelike*,
- if $U^\mu U_\mu < 0$, the vector is said to be *timelike*,
- if $U^\mu U_\mu = 0$, the vector is said to be *null*.

1.2.4 Proper time and locally inertial frames

In relativistic theories, one postulates that particles with zero mass follow curves on $\mathcal{M}$ for which tangent vectors are null, and massive particles (point masses) are said to follow *worldlines*: curves for which all tangent vectors are timelike. At every point of a spacetime, it is therefore possible to define a local lightcone and any worldline passing through this point should lie within the lightcone.

For point masses following worldlines, one defines their *proper time* τ first through the infinitesimal change along a worldline, from $x^\mu(\lambda)$ (λ being again a parameter along the worldline) to $x^\mu + dx^\mu(\lambda + d\lambda)$. The square of the infinitesimal variation of the point mass proper time is given by:

$$d\tau^2 = -\frac{1}{c^2}ds^2 = -\frac{1}{c^2}g_{\mu\nu}dx^\mu dx^\nu. \qquad (1.19)$$

The time along the worldline is obtained integrating the square root of this expression; it is the time measured by a clock moving along this worldline.

Thus, to every point mass moving along a worldline is associated the vector field of the 4-velocity u^μ

$$u^\mu = \frac{1}{c}\frac{dx^\mu}{d\tau}, \qquad (1.20)$$

and with the definition of proper time (1.19), one sees that u^μ is a timelike vector, with the constant norm:

$$u^\mu u_\mu = -1. \qquad (1.21)$$

To every worldline can be associated an *observer*, whose 4-velocity is thus defined too.

Let $g_{\mu\nu}(x^\rho)$ be the components of the metric tensor in a given coordinate system. In another system $X^\sigma(x^\rho)$ the components of this tensor shall be $G_{\mu\nu}(X^\rho)$, computed from $g_{\mu\nu}$ and the Eq. (1.13) for the change of coordinates. If we now make a Taylor expansion around a point $P_0(X_0) = P_0(x_0)$:

$$\begin{aligned} G_{\mu\nu}(X) \;=\; & \frac{\partial x^\rho}{\partial X^\mu}\frac{\partial x^\sigma}{\partial X^\nu}g_{\rho\sigma}(x_0) + (X^\alpha - X_0^\alpha)\left(g_{\rho\sigma}\frac{\partial^2 x^\rho}{\partial X^\sigma \partial X^\mu}\frac{\partial x^\sigma}{\partial X^\nu}\right. \\ & \left. + g_{\rho\sigma}\frac{\partial^2 x^\rho}{\partial X^\sigma \partial X^\nu}\frac{\partial x^\sigma}{\partial X^\mu} + \frac{\partial x^\rho}{\partial X^\mu}\frac{\partial x^\sigma}{\partial X^\nu}\frac{\partial g_{\rho\sigma}}{\partial X^\alpha}\right)(x_0) + \mathcal{O}\left(X^\alpha - X_0^\alpha\right)^2 \end{aligned} \qquad (1.22)$$

Can one devise a change of coordinates such that

$$G_{\mu\nu}(X) = \eta_{\mu\nu}(x_0) + \mathcal{O}\left(X^\alpha - X_0^\alpha\right)^2? \tag{1.23}$$

Given that there are 10 components of the metric $g_{\rho\sigma}(x_0)$ and 40 components for its first derivatives $\partial_\alpha g_{\rho\sigma}$ on the one hand, 16 numbers $\dfrac{\partial x^\rho}{\partial x^\mu}$ and 40 second derivatives for the coordinate change on the other hand, it is possible to get a solution and have locally the Minkowski metric as in (1.23). It is thus always possible to make a **local** change of coordinates so that the metric be that of a flat spacetime up to second-order terms. These terms cannot be set to zero by a suitable change of coordinates and they represent curvature effects, as described by the Riemann tensor (see Sec. 1.3.1). Such coordinates correspond to *local inertial frames* and are a direct application of the equivalence principle.

1.2.5 Geodesics

The equations for the worldlines of free particles on the manifold (only in presence of gravitation) can be naively derived taking a locally inertial frame, with coordinates $\{X^\mu\}$ and writing that the worldline equation verifies

$$\frac{d^2 X^\mu}{d\lambda^2} = 0, \tag{1.24}$$

λ here can be taken as the proper time for a massive particle, or be a parameter. Taking a general frame $\{x^\mu\}$, one obtains the equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0, \tag{1.25}$$

where the quantities

$$\Gamma^\mu_{\nu\rho} = \frac{\partial x^\mu}{\partial X^\sigma}\frac{\partial^2 X^\sigma}{\partial x^\nu \partial x^\rho} \tag{1.26}$$

are called the *Christoffel symbols* and are symmetric in the ν and ρ indices.

Equation (1.25) defines the *geodesic* for a particle. They are defined as the curves that make extremal the distance between two points P and Q on the manifold. In the case of a massive particle:

$$\delta \int_P^Q d\tau = 0, \tag{1.27}$$

with $d\tau$ the proper time defined by Eq. (1.19), leads after a few lines of calculation to the same Eq. (1.25), with the expression for the Christoffel symbols:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(\frac{\partial g_{\nu\sigma}}{\partial x^\rho} + \frac{\partial g_{\sigma\rho}}{\partial x^\nu} - \frac{\partial g_{\nu\rho}}{\partial x^\sigma}\right). \tag{1.28}$$

In a locally inertial frame, one has that all Christoffel symbols vanish, as they only depend on first derivatives of the metric. This shows that they are not tensors, since otherwise they would be zero in any frame, thanks to the definition of a tensor (1.13).

1.2.6 Covariant derivative

This raises the point on being careful that "everything with indices" is not in general a tensor. What about derivatives of tensors? In the case of the gradient of a scalar field $\frac{\partial S}{\partial x^\mu}$, its transformation under a change of coordinates shows that it verifies the definition of a form (1.12).

For the gradient of a higher-order tensor, this is not the case. A first problem comes from the fact that, when evaluating the (infinitesimal) difference between two vector at two different points, one has to deal with objects belonging to two different spaces, since each point P has attached to it a different tangent space $\mathcal{T}_P$.

Still, it is possible to define a derivative operator D_α that satisfies the usual properties for a derivation (linearity, Leibnitz rule, ...) and which is supposed to transform a $\begin{pmatrix} p \\ q \end{pmatrix}$ tensor into a $\begin{pmatrix} p \\ q+1 \end{pmatrix}$ one. Once a base $\{\vec{e}_\mu\}$ for the tangent space is chosen, one can write down the action of D_α on a vector field, and one gets

$$D_\alpha \vec{v} = D_\alpha \left(v^\mu \vec{e}_\mu \right) = \frac{\partial v^\mu}{\partial x^\alpha} \, \vec{e}_\mu + v^\mu D_\alpha \vec{e}_\mu. \tag{1.29}$$

Actually, to specify this *covariant derivative*, one needs to set the 64 *connection coefficients*:

$$\gamma^\nu_{\mu\alpha} \text{ such that } D_\alpha \vec{e}_\mu = \gamma^\nu_{\mu\alpha} \vec{e}_\nu. \tag{1.30}$$

Then, it is easy to get the formula for forms:

$$D_\alpha W_\mu = \frac{\partial W_\mu}{\partial x^\alpha} \, \underline{e}_\mu - \gamma^\nu_{\mu\alpha} W_\nu, \tag{1.31}$$

and therefore for any type of tensor

$$\begin{aligned} D_\alpha T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} &= \frac{\partial}{\partial x^\rho} T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} \vec{e}_{\mu_1} \otimes \cdots \otimes \vec{e}_{\mu_p} \otimes \underline{e}_{\nu_1} \cdots \otimes \underline{e}_{\nu_q} \\ &+ \sum_{r=1}^{p} \gamma^{\mu_r}_{\sigma\rho} \, T^{\mu_1 \dots \mu_{r-1} \sigma \dots \mu_p}{}_{\nu_1 \dots \nu_q} \\ &- \sum_{r=1}^{q} \gamma^{\sigma}_{\nu_r\rho} \, T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_{r-1} \sigma \dots \nu_q}. \end{aligned} \tag{1.32}$$

The most appropriate choice for the connection coefficients is to take the Christoffel symbols defined by (1.28):

$$\gamma^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho}. \tag{1.33}$$

This is called a Riemannian connection, or Levi-Civita connection. Two important consequences of the covariant derivative thus defined are:

- The second derivatives of a scalar field commute: $D_\alpha D_\beta S = D_\beta D_\alpha S$; this is due to the symmetry in the lower indices of the Christoffel symbols. The connection is said to have no torsion.

- The covariant derivative of the metric tensor is zero

$$D_\alpha g_{\mu\nu} = 0. \tag{1.34}$$

  The connection is said to be compatible with the metric.

Finally, let us mention here the *Lie derivative*, which is a very "simple" derivative that does not need any metric to be defined on the manifold. It can be seen as the derivative of a tensor field along the directions given by a vector field, *e.g.* the Lie derivative of a vector field U^μ along the field V^ν is given by

$$\mathcal{L}_{\vec{V}} U^\mu = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu, \tag{1.35}$$

where the notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

has been used. An interesting feature of the Lie derivative is that it can be defined using partial derivatives, as in Eq. (1.35), or with the covariant derivative

$$\mathcal{L}_{\vec{V}} U^\mu = V^\nu D_\nu U^\mu - U^\nu D_\nu V^\mu,$$

which gives the same result.

1.3 Riemann, Ricci, Weyl (tensors) and Einstein equations

1.3.1 Riemann tensor

As it has been shown previously in Sec. 1.2.6, the second covariant derivative acting on a scalar field can commute. However, this is not true for a higher-order tensor fields and in particular, vectors. In that case, the commutator reads (Ricci identity):

$$D_\mu D_\nu V^\rho - D_\nu D_\mu V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma. \tag{1.36}$$

$R^\rho{}_{\sigma\mu\nu}$ is a $\binom{1}{3}$ tensor, called the *Riemann tensor*. Its tensorial nature comes from this definition, as the covariant derivative of a tensor is a tensor. The explicit formula to compute the Riemann tensor is

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\alpha_{\sigma\nu}\Gamma^\rho_{\alpha\mu} - \Gamma^\alpha_{\sigma\mu}\Gamma^\rho_{\nu\alpha}. \tag{1.37}$$

Among many interpretation of the Riemann tensor, let us mention here two:

- If one considers a vector V_0^μ, which is transported parallel to itself (using the connection compatible with the metric) using two infinitesimal paths $dx_1^\mu \to dx_2^\mu$, and the inverse order $dx_2^\mu \to dx_1^\mu$, the result V_1^μ shall be different depending on the order.

$$V_1^\mu(dx_1^\mu \to dx_2^\mu) - V_1^\mu(dx_2^\mu \to dx_1^\mu) = R^\mu{}_{\nu\sigma\rho} V_0^\nu dx_1^\sigma dx_2^\rho.$$

  This is the indication that the connection which ensuring the parallel transport is curved.

- When one looks at two infinitesimally close geodesics: one described by $x^\mu(\lambda)$ and the other by $x^\mu(\lambda) + \delta x^\mu(\lambda)$, the difference being called *geodesic deviation*. In flat (Minkowski) spacetime, this deviation is a linear function of the parameter λ. Writing the geodesic equation for each curve:

$$\begin{aligned} \frac{d^2 x^\mu}{d\lambda^2} &+ \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda}\frac{dx^\rho}{d\lambda} = 0, \\ \frac{d^2(x^\mu + \delta x^\mu)}{d\lambda^2} &+ \Gamma^\mu_{\sigma\rho}(x+\delta x)\frac{d(x^\sigma + \delta x^\sigma)}{d\lambda}\frac{d(x^\rho + \delta x^\rho)}{d\lambda} = 0, \end{aligned}$$

  and developing the difference at first order in δx:

$$\frac{d^2}{d\lambda^2}\delta x^\mu = R^\mu{}_{\nu\rho\sigma} \frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}\delta x^\sigma. \tag{1.38}$$

  This equation gives the relative deviation between two free falling particles in a gravitational field. The presence of the Riemann tensor shows the influence of the gravitational field and indicates that "gravitational forces" are better expressed in terms of this tensor than in terms of the metric or the Christoffel symbols.

From the definition (1.36) or from the above two illustration, one can see that

$$\text{flat (Minkowski) spacetime} \iff R_{\mu\nu\sigma\rho} = 0, \tag{1.39}$$

the first index being lowered by contraction with the metric tensor.

The Riemann tensor fulfills some algebraic and differential identities:

- It is antisymmetric in the first and last pair of indices:

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}.$$

- It is symmetric in the exchange of first and last pair of indices:

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}.$$

- It possesses a cyclic symmetry with respect to the last three indices:

$$R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\sigma\nu\rho} + R^\mu{}_{\rho\sigma\nu} = 0.$$

These properties reduce the number of independent components of the Riemann tensor to 20. The fundamental differential identity is called the *Bianchi identity*:

$$D_\alpha R^\mu{}_{\nu\rho\sigma} + D_\rho R^\mu{}_{\nu\sigma\alpha} + D_\sigma R^\mu{}_{\nu\alpha\rho} = 0. \tag{1.40}$$

1.3.2 Ricci and Einstein tensors

From the Riemann tensor it is possible to define several useful tensors, as the *Ricci tensor* from the contraction

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}. \tag{1.41}$$

The Ricci tensor $R_{\mu\nu}$ is symmetric and it appears to be the only second-order tensor that can be obtained by contraction of the Riemann tensor; other contraction lead to $\pm R_{\mu\nu}$ or 0. Then, the scalar

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu\nu}_{\mu\nu}, \tag{1.42}$$

is called the *Ricci scalar* or scalar curvature. It is the only non-zero scalar field that one can obtain from the Riemann tensor.

Finally, the Bianchi identities (1.40) when contracted on the first and last indices, on the one hand, and the second and third on the other hand give

$$D_\alpha \left(R^{\alpha\mu} - \frac{1}{2} g^{\alpha\mu} R \right) = 0. \tag{1.43}$$

The tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{1.44}$$

is called the *Einstein tensor* and plays a central role in the Einstein equations. Note that the conditions $R_{\mu\nu} = 0$, $R = 0$ or $G_{\mu\nu} = 0$ do not mean that the spacetime is flat.

1.3.3 Weyl tensor

What is left in the Riemann tensor that is not contained in the Ricci tensor enters in the so-called *Weyl tensor*

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - (g_{\mu\rho}R_{\sigma\nu} - g_{\mu\sigma}R_{\rho\nu} - g_{\nu\rho}R_{\sigma\mu} + g_{\nu\sigma}R_{\rho\mu}) + \frac{1}{3}\left(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu}\right) R. \tag{1.45}$$

The Weyl tensor has the same symmetries as the Riemann tensor, and moreover it is traceless

$$C^{\mu}{}_{\rho\mu\sigma} = 0.$$

The Weyl tensor has 10 independent components and, if the Ricci tensor is zero (as it is the case when the Einstein equation hold in vacuum), then the Weyl and the Riemann tensors coincide.

We now rapidly introduce the Newman-Penrose formalism to obtain a better description of the Weyl tensor. The basic idea is to introduce a tetrad of null vectors. Let us first start from an orthonormal tetrad $\left\{\vec{e}_{(\alpha)}\right\}$, so that the metric becomes

$$g_{\mu\nu} = -e_{(0)\mu}e_{(0)\nu} + e_{(1)\mu}e_{(1)\nu} + e_{(2)\mu}e_{(2)\nu} + e_{(3)\mu}e_{(3)\nu}.$$

We can choose $e^\mu_{(0)}$ as a unit vector along $\vec{\partial}_t$ (see Sec. 1.4), $e^\mu_{(1)}$ as the unit radial vector in spherical coordinates and $\left(e^\mu_{(2)}, e^\mu_{(3)}\right)$ as unit vectors in the angular directions. We then build the null tetrad from the **complex** vectors

$$\begin{aligned} l^\mu &= \frac{1}{\sqrt{2}}\left(e^\mu_{(0)} + e^\mu_{(1)}\right), \\ k^\mu &= \frac{1}{\sqrt{2}}\left(e^\mu_{(0)} - e^\mu_{(1)}\right), \\ m^\mu &= \frac{1}{\sqrt{2}}\left(e^\mu_{(2)} + ie^\mu_{(1)}\right), \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}}\left(e^\mu_{(2)} - ie^\mu_{(1)}\right). \end{aligned}$$

The 10 components of the Weyl tensors can then be represented by the 5 complex scalars, called *Weyl scalars*:

$$\begin{aligned} \Psi_0 &= C_{\mu\nu\rho\sigma} l^\mu m^\nu l^\rho m^\sigma, \\ \Psi_1 &= C_{\mu\nu\rho\sigma} l^\mu k^\nu l^\rho m^\sigma, \\ \Psi_2 &= C_{\mu\nu\rho\sigma} l^\mu m^\nu \bar{m}^\rho k^\sigma, \\ \Psi_3 &= C_{\mu\nu\rho\sigma} l^\mu k^\nu \bar{m}^\rho k^\sigma, \\ \Psi_4 &= C_{\mu\nu\rho\sigma} k^\mu \bar{m}^\nu k^\rho \bar{m}^\sigma. \end{aligned} \tag{1.46}$$

Gravitational radiation content of the spacetime can conveniently be described using some of the Weyl scalars (see Sec. 2.3.2).

1.3.4 Stress-energy tensor

At this point, we have to specify how "matter" enters the theory of general relativity. To allow for the most general case, it is more convenient to introduce it from its Lagrangian through the *stress-energy tensor* $T_{\mu\nu}$. Given a Lagrangian L for a matter model, the stress-energy tensor is defined as:

$$T_{\mu\nu} = -\frac{\partial L}{\partial g^{\mu\nu}} + \frac{g_{\mu\nu}}{2} L. \tag{1.47}$$

If one considers the action S

$$S = \int_\Omega L\sqrt{-g}\, d^4x,$$

where $g = \det g_{\mu\nu}$ is the determinant of the metric; and its variation with respect to the metric $\delta g_{\mu\nu}$, it is possible to show (after some integration) that the stress-energy tensor should be divergence-free:

$$D_\mu T^{\mu\nu} = 0. \tag{1.48}$$

As an illustration of the role of this tensor, let us consider an observer with 4-velocity u^μ_0:

- the energy density measured by this observer

$$\epsilon = T_{\mu\nu}\, u_0^\mu\, u_0^\nu,$$

- the 3-vector of linear momentum, as measured by this observer, along the direction e_i^μ (normal to the direction given by u_0^μ)

$$p^i = -\frac{1}{c} T_{\mu\,\nu}\, e_i^\mu\, u_0^\nu,$$

and $p^\mu = p^i e_i^\mu$. With these definitions, the stress-energy tensor is sadi to satisfy the *weak energy condition* if $\epsilon \geq 0$ for any observer. Furthermore, if $p^\mu p_\mu c^2 \leq \epsilon$ then matter is said to satisfy the dominant energy condition.

From Eq.(1.47), one can easy get a stress-energy tensor for a given model for matter. However, if matter is phenomenologically described as a perfect fluid, then the tensor is

$$T^{\mu\nu} = (\epsilon + p)\; u^\mu u^\nu + p\, g^{\mu\nu}, \tag{1.49}$$

where ϵ is the energy density, p the pressure (both measured in the fluid frame), and u^μ its 4-velocity.

1.3.5 Einstein equations

We have now gathered all objects to give the Einstein equations. Intuitively, they relate the curvature (Riemann tensor) to the matter content (stress-energy tensor) in a covariant relation, with the correct Newtonian limit, *i.e.* Newton's law (1.4).

Let us start with the expression

$$K_{\mu\nu} = \chi T_{\mu\nu}, \tag{1.50}$$

where $K_{\mu\nu}$ and χ are a tensor and constant to be determined. As $T_{\mu\nu}$ is a symmetric, divergence-free tensor, so must be $K_{\mu\nu}$. The most general one obtained from the Riemann tensor is (see Sec. 1.3.2):

$$K_{\mu\nu} = R_{\mu\nu} + aR\, g_{\mu\nu} + \Lambda g_{\mu\nu}.$$

From the divergence-free condition and Eq. (1.43), $a = -\dfrac{1}{2}$ and one obtains:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \chi T_{\mu\nu}. \tag{1.51}$$

Λ is know as the *cosmological constant* and is negligible, as long as one does not consider cosmological evolution (or evolution of a large part of visible Universe). We shall therefore neglect it hereafter. If we now take the "non-relativistic" limit (*i.e.* taking $c \to +\infty$) of this equation with a perfect fluid model for the stress energy tensor, we get (after a few lines of algebra):

$$\Delta g_{00} = \chi\epsilon = \chi\rho c^2,$$

Where ρ is the mass density. On the other hand, a Newtonian limit on g_{00} gives:

$$g_{00} \simeq -1 + \frac{2\Phi}{c^2}, \tag{1.52}$$

with Φ the Newtonian potential of Eq. (1.4). So that $\chi = \dfrac{8\pi G}{c^4}$ and the Einstein equations are:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1.53}$$

There are many other ways of presenting these equations, among all the possibilities, let us mention the variational approach due to Hilbert. The idea here is to deduce the Einstein equations (1.53) from the extremization of the action

$$\delta S = 0 = \delta \int \sqrt{-g}\, d^4x \left(\mathcal{L}_{\text{grav.}} + \mathcal{L}_{\text{mat.}}\right).$$

The term $\sqrt{-g}$ has been introduced so that the volume element be invariant under a coordinate change:

$$\sqrt{-g}\, d^4x = \sqrt{-g'} d^4x',$$

and $\mathcal{L}_{\text{grav.}}$ and $\mathcal{L}_{\text{mat.}}$ are the Lagrangian for gravitation and matter, respectively. To determine $\mathcal{L}_{\text{grav.}}$, one can take the simplest scalar that can be formed from the curvature, namely:

$$\mathcal{L}_{\text{grav.}} = \text{const.} \times R. \tag{1.54}$$

Varying the action with respect to the metric $g_{\mu\nu}$ and the connection $\Gamma^{\mu}_{\nu\rho}$, one obtains (after some work) the same expression (1.53), with the constant determined (again) from the Newtonian limit.

1.4 Introduction to 3+1 formalism

1.4.1 Introduction to the introduction...

The gauge freedom, together with the "general" mixing of space and time may be a problem to prove some general mathematical results as the well-posedness of the equations. In particular, it is more convenient to have a formalism in which Einstein equations can be cast into a Cauchy problem: given some initial data, how does the evolution in time behave? The *3+1 formalism* is such an approach that considers the slicing of the four-dimensional manifold $\mathcal{M}$ by spacelike three-dimensional surfaces. The induced metric on these hypersurfaces is then of signature $(+, +, +)$, and the remaining coordinate is "the time", which is labeling the hypersurfaces. Although this decomposition in "space" and "time" is not unique, it helps a lot in having a more standard formulation, with Riemannian scalar product, 3-vectors and 3-tensors on the hypersurfaces.

Historically, this formalism has been developed since the 1920's by G.Darmois, and later by A.Lichnerowicz and Y.Choquet-Bruhat. In the 1960's, the 3+1 formalism served as a foundation to the Hamiltonian formulation of general relativity, by P.A.M Dirac and R.Arnowitt, S.Deser and C.Misner.[2] More recently, the numerical relativity community has made an extensive use of 3+1 formalism for obtaining numerical solutions of Einstein equations.

Under the condition of global hyperbolicity (that there exists a spacelike hypersurface such that every timelike or null curve without an end point intersects it exactly once), it is possible to foliate the spacetime $(\mathcal{M}, g_{\mu\nu})$ by a family of spacelike hypersurfaces. This means that one can find a smooth scalar field t, such that each hypersurface is a level surface of this field, that we note Σ_t. We have the properties:

$$\begin{aligned} \text{if } t_1 \neq t_2, \qquad & \Sigma_{t_1} \cap \Sigma_{t_2} = \emptyset, \\ \text{and} \qquad & \mathcal{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t. \end{aligned} \tag{1.55}$$

The vector field $D^\mu t$ is timelike and defines the unique normal direction to all the Σ_t's. It can be normalized, so that we define

$$n^\mu = -N D^\mu t, \quad \text{with } N = \frac{1}{\sqrt{-D^\mu t\, D_\mu t}}. \tag{1.56}$$

n^μ is the future-directed unit vector normal to the slice Σ_t and N is called the *lapse function* (in many studies, it is noted α). With the unitarity property of n^μ, it is possible to associate an observer to this vector field, which is the regarded as a 4-velocity. The observer is called *Eulerian observer*.

1.4.2 Fundamental forms

The 3-metric induced on each hypersurface Σ_t by the global metric $g_{\mu\nu}$ measures the proper distances on this surface[3]:

$$dl^2 = \gamma_{ij}\, dx^i\, dx^j, \tag{1.57}$$

it is also called the *first fundamental form* on Σ_t. The third (after the lapse N and γ_{ij}) basic ingredient to describe the full spacetime is the *shift vector* β^i, which measures the relative velocity between the Eulerian observer and lines of constant spatial coordinates.

In terms of these quantities the spacetime metric takes the form:

$$ds^2 = \left(-N^2 + \beta^i \beta_i\right) dt^2 + 2\beta_i\, dt\, dx^i + \gamma_{ij}\, dx^i dx^j, \tag{1.58}$$

where one can raise and lower Latin indices using the 3-metric: $\beta_i = \gamma_{ij}\beta^j$. The four-dimensional volume element is given by

$$\sqrt{-g} = N\sqrt{\gamma}, \tag{1.59}$$

[2] their initials (ADM) are often used to denote the 3+1 formalism, although they were not the first to design it.

[3] remember that Latin indices range from 1 to 3

where γ is the determinant of the 3-metric. Finally, the components of the unit normal vector n^μ are given by:

$$n^\mu = \left(\frac{1}{N}, -\frac{\beta^1}{N}, -\frac{\beta^2}{N}, -\frac{\beta^2}{N}\right). \tag{1.60}$$

The curvature tensor associated to the 3-metric ${}^{(3)}R^i{}_{jkl}$ measures the intrinsic curvature of each Σ_t. The *extrinsic curvature* describes the way in which those hypersurfaces are embedded in the four-dimensional spacetime. It is defined from the variation of the normal unit vector, when transported from one point of the hypersurface to another. It is defined as

$$K_{\mu\nu} = -P^\rho{}_\mu D_\rho n_\nu, \tag{1.61}$$

with $P^\rho{}_\mu$ the projection operator on Σ_t. Another equivalent definition is that the extrinsic curvature is the Lie derivative along the normal direction of the spatial metric

$$K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_{\vec{n}}\,\gamma_{\mu\nu}. \tag{1.62}$$

From this expression one can deduce that the extrinsic curvature is tangent to the hypersurface Σ_t and symmetric with respect to its two indices. A relation giving the extrinsic curvature in terms of the (3+1 decomposed) metric is:

$$K_{ij} = \frac{1}{2N}\left(\nabla_i\beta_j + \nabla_j\beta_i - \frac{\partial\gamma_{ij}}{\partial t}\right), \tag{1.63}$$

where ∇_i is the covariant derivative compatible with the 3-metric γ_{ij}. The extrinsic curvature is also called the *second fundamental form*.

1.4.3 Projection of the Einstein equations

The metric being projected onto the Σ_t's and along their normal, it is now interesting to see how do the Einstein equations (1.53) translate into the 3+1 variables. First, let us decompose the stress-energy tensor $T_{\mu\nu}$ along n^μ worldlines and Σ_t. The matter energy density, as measured by the Eulerian observer is

$$E = T_{\mu\nu}\, n^\mu n^\nu, \tag{1.64}$$

and similarly, the matter momentum density (which is tangent to Σ_t):

$$J_\mu = -T_{\nu\rho}\, n^\nu \gamma^\rho{}_\mu. \tag{1.65}$$

Finally, the matter stress tensor is the tensor field tangent to Σ_t too:

$$S_{\mu\nu} = T_{\rho\sigma}\, \gamma^\rho{}_\mu\, \gamma^\sigma{}_\nu. \tag{1.66}$$

As these two tensors are tangent to Σ_t, we can write only their spatial components: J_i and S_{ij}. In particular, the trace S is given by:

$$S = \gamma^{ij}\, S_{ij}. \tag{1.67}$$

The details of the calculations giving the projected Einstein equation shall not be given here, but only the results shall be listed. Note however, that extensive use is made of Gauss and Codazzi equations (not given here). When projecting twice on Σ_t, one obtains an evolution equation for the extrinsic curvature:

$$\begin{aligned} \frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\vec{\beta}} K_{ij} &= -\nabla_i \nabla_j N \\ &+ N \left\{ {}^{(3)}R_{ij} + K\, K_{ij} - 2K_{ik}K^k_{\ j} + \frac{4\pi G}{c^4} \left[(S-E)\,\gamma_{ij} - 2S_{ij} \right] \right\}. \end{aligned} \tag{1.68}$$

Recalling that K_{ij} is linked to the time-derivative of γ_{ij}, this is a second-order in time equation for the 3-metric.

Projecting twice onto the normal to Σ_t, one has:

$$ {}^{(3)}R + K^2 - K_{ij}K^{ij} = \frac{16\pi G}{c^4} E, \tag{1.69}$$

which is called the *Hamiltonian constraint* and is an elliptic-type partial differential equation. It means that it does not describe any propagation, but is more similar (although non-linear) to the Poisson equation (1.4). Finally, projecting once onto Σ_t and once along the normal n^μ, one obtains the *momentum constraint*:

$$\nabla_j K^j_{\ i} - \nabla_i K = \frac{8\pi G}{c^4} J_i, \tag{1.70}$$

which is an elliptic-type partial differential equation for 3-vectors.

1.4.4 Weyl electric and magnetic tensors

With the unit vector n^μ, it is possible to define the electric and magnetic parts of the Weyl tensor (1.45):

$$E_{\mu\nu} = n^\rho\, n^\sigma\, C_{\rho\mu\sigma\nu}, \tag{1.71}$$

$$B_{\mu\nu} = \frac{1}{2} n^\rho\, n^\sigma\, C_{\rho\mu\alpha\beta}\, \varepsilon^{\alpha\beta}_{\ \ \sigma\nu}, \tag{1.72}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita completely antisymmetric tensor. The symmetries of the Weyl tensor imply that these two tensors are both symmetric, traceless and tangent to Σ_t.

Using again decomposition of the 4-Riemann tensor into 3+1 quantities, it is possible to write electric and magnetic Weyl tensors in 3+1 language as:

$$E_{ij} = R_{ij} + K\, K_{ij} - K_{im}\, K^m_{\ j} - \frac{4\pi G}{c^4} \left[S_{ij} + \frac{1}{3}\gamma_{ij}\,(4E - S) \right], \tag{1.73}$$

$$B_{ij} = \varepsilon_i^{\ mn} \left(\nabla_m K_{nj} - \frac{4\pi G}{c^4} \gamma_{jm}\, J_n \right), \tag{1.74}$$

with ε_{ijk} being now the Levi-Civita tensor in three dimensions.

Chapter 2

Gravitational Waves and Astrophysical Solutions

2.1 Spherical symmetry and Schwarzschild solution

2.1.1 Spherically symmetric spacetime

We here give a solution to the Einstein equations (1.53) in the "spherically symmetric" case. The notion of symmetry on a manifold needs some more clarifications, with the definition of a *Killing vector field*. A spacetime is said to possess a symmetry if the metric is invariant under the Lie derivative (1.35) with respect to some vector field ξ^μ:

$$\mathcal{L}_{\vec{\xi}}\, g_{\mu\nu} = 0; \tag{2.1}$$

$\vec{\xi}$ is then called a *Killing field*. If one takes $\vec{\xi} = \vec{\partial}_1$ (associated to the coordinate x^1), then the consequence of E.q(2.1) is that the metic does not depend on this coordinate. As discussed at the end of Sec. 1.2.6, one can use the covariant derivative to express the Lie derivative. In that case and using the Ricci theorem (1.34), Eq. (2.1) translates into

$$\mathcal{L}_{\vec{\xi}}\, g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu = 0, \tag{2.2}$$

which is called the *Killing equation*.

Considering now coordinates of the spherical type (t, r, θ, φ) (the t coordinate being eventually defined from a 3+1 split, see Sec. 1.4). The notion of spherical symmetry comes from the existence of three spacelike Killing fields:

- $\vec{\xi}_{(z)} = \vec{\partial}_\varphi$, for the symmetry with respect to the z-axis,
- $\vec{\xi}_{(x)} = -\sin\varphi\, \vec{\partial}_\theta - \cot\theta \cos\varphi\, \vec{\partial}_\varphi$ for the symmetry with respect to the x-axis,
- $\vec{\xi}_{(y)} = -\cos\varphi\, \vec{\partial}_\theta - \cot\theta \sin\varphi\, \vec{\partial}_\varphi$ for the symmetry with respect to the y-axis.

Within the existence of these four Killing fields, the most general spherically symmetric spacetime can write:

$$ds^2 = -B(r,t)\, c^2 dt^2 + A(r,t)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{2.3}$$

2.1.2 Schwarzschild metric

We here look for the solution of Einstein equations (1.53) in the case of vacuum ($T^{\mu\nu} = 0$) and spherically symmetric spacetime. Contracting the Einstein equations, one gets in the vacuum case:

$$R - \frac{1}{2} R\, g^{\mu\nu} g_{\mu\nu} = 0,$$

which means that, in the vacuum case the scalar curvature is zero. It is thus sufficient to solve

$$R_{\mu\nu} = 0.$$

From the line element (2.3), one can compute the Christoffel symbols using Eq. (1.28) to obtain

$$\begin{aligned}
\Gamma^t_{tt} &= \frac{\dot{B}}{2B}, \quad \Gamma^t_{tr} = \Gamma^t_{rt} = \frac{B'}{2B}, \quad \Gamma^t_{rr} = \frac{\dot{A}}{2B}, \\
\Gamma^r_{tt} &= \frac{B'}{2A}, \quad \Gamma^r_{tr} = \Gamma^r_{rt} = \frac{\dot{A}}{2A}, \quad \Gamma^r_{rr} = \frac{A'}{2A}, \quad \Gamma^r_{\theta\theta} = -\frac{r}{A}, \quad \Gamma^r_{\varphi\varphi} = -\frac{r\sin^2\theta}{A}, \\
\Gamma^\theta_{r\theta} &= \Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \\
\Gamma^\varphi_{r\varphi} &= \Gamma^\varphi_{\varphi r} = \frac{1}{r}, \quad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{\cos\theta}{\sin\theta},
\end{aligned} \tag{2.4}$$

with a dot $\dot{}$ and the prime $'$ denoting derivatives with respect to the coordinate t and r respectively. All the other Christoffel symbols are zero.

To compute the Ricci tensor, one has to take the formula giving the Riemann tensor (1.37), contracting it as in the definition of the Ricci tensor (1.41), to obtain

$$\begin{aligned}
R_{tt} &= -\frac{\ddot{A}}{2A} + \frac{\dot{A}^2}{4A^2} + \frac{\dot{B}\dot{A}}{4AB} + \frac{B''}{2A} - \frac{B'A'}{4A^2} + \frac{B'}{Ar} - \frac{B'^2}{4AB} = 0, \\
R_{rr} &= \frac{\ddot{A}}{2B} - \frac{\dot{B}\dot{A}}{4B^2} - \frac{\dot{A}^2}{4AB} + \frac{A'}{Ar} - \frac{B''}{2B} + \frac{B'^2}{4B^2} + \frac{A'B'}{4AB} = 0, \\
R_{tr} &= R_{rt} = \frac{\dot{A}}{Ar} = 0, \\
R_{\theta\theta} &= 1 - \frac{1}{A} + \frac{rA'}{2A^2} - \frac{rB'}{2AB} = 0, \\
R_{\varphi\varphi} &= \sin^2\theta\, R_{\theta\theta} = 0,
\end{aligned} \tag{2.5}$$

the other components being null. From the R_{rt} equation, one deduces that A does not depend on t, so the Einstein equations reduce to the following system:

$$\begin{aligned}
-\frac{B''}{2A} + \frac{B'A'}{4A^2} - \frac{B'}{Ar} + \frac{B'^2}{4AB} &= 0, \\
\frac{B''}{2B} - \frac{B'^2}{4B^2} - \frac{A'B'}{4AB} - \frac{A'}{Ar} &= 0, \\
-1 + \frac{1}{A} - \frac{rA'}{2A^2} + \frac{rB'}{2AB} &= 0,
\end{aligned}$$

which is equivalent to

$$
\begin{aligned}
(AB)' &= 0, && (2.6) \\
\left(\frac{r}{A}\right)' &= 1. && (2.7)
\end{aligned}
$$

The general solution is

$$A = \frac{1}{1 - \frac{\kappa}{r}}, \qquad B = f(t)\left(1 - \frac{\kappa}{r}\right),$$

where κ is an integration constant and $f(t)$ an arbitrary function. This function can be set to 1 with an appropriate change of coordinates $t \to t'$, such that $dt' = \sqrt{f(t)}\, dt$. the constant κ can be determined by taking the limit for $r \to +\infty$ for the geodesics in this metric. One then recovers a Keplerian motion around a body of mass M so that $\kappa = \dfrac{2GM}{c^2}$. Finally the metric is

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \frac{1}{1 - \frac{2GM}{rc^2}} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \qquad (2.8)$$

and is called *Schwarzschild metric*.

It appears that this most general metric in spherical symmetry is static too,[1] This is actually the *Birkhoff theorem*: the exterior gravitational field of spherically symmetric matter distribution is static (and given by the Schwarzschild solution). It is true in particular for the metric outside a spherically collapsing or oscillating body. The Schwarzschild metric is also *asymptotically flat*:

$$\lim_{r \to +\infty} g_{\mu\nu} = \eta_{\mu\nu}, \qquad (2.9)$$

it tends toward the Minkowski metric at spatial infinity. This is a general property for metrics describing isolated systems, but is usually not the case in cosmology.

2.1.3 Black holes

The Schwarzschild solution as given by Eq. (2.8) possesses two singularities: at $r = 0$ and

$$r = \frac{2GM}{c^2} = R_S > 0, \qquad (2.10)$$

which is called the *Schwarzschild radius* of the central object. For ordinary stars the Schwarzschild radius is much smaller than the actual radius (for the Sun $R_S \simeq 3$ km). It is sometimes relevant to compare the radius R of a star to its Schwarzschild radius:

$$\Xi = \frac{GM}{Rc^2}, \qquad (2.11)$$

[1] Spacetime is said to be *stationary* if $\vec{\partial}_t$ is a Killing field; it is called *static* if the vectors $\vec{\partial}_t$ are perpendicular to the hypersurfaces $t =$ const. (the shift $\beta^i = 0$ in the Σ_t hypersurfaces in Sec. 1.4).

and one defines a *compact object* to be an object for which $\Xi \geq 10^{-4}$. One has $\Xi = 0.5$ by definition for a *black hole*.

Coming back the the singularity at $r = R_S$ in the Schwarzschild metric, a way of seeing whether it is a real singularity (some physical observable diverge) or a coordinate singularity (as at $r = 0$ of polar coordinate system), is to try to find some new coordinates in which the problems would disappear. One such a choice are the so-called *3+1 Eddington-Finkelstein coordinates* obtained by changing only the time coordinate:

$$\hat{t} = t + \frac{R_S}{c} \ln\left(\frac{r}{R_S} - 1\right), \tag{2.12}$$

for which the metric changes to:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 d\hat{t}^2 + \frac{2GM}{rc} d\hat{t}\, dr + \left(1 + \frac{2GM}{rc^2}\right) dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{2.13}$$

The components of this metric are regular at $r = R_S$, showing that this was a coordinate singularity. On the contrary the singularity at $r = 0$ is a real one, as can be seen by computing the scalar obtained contracting the Riemann tensor with itself[2]:

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48 G^2 M^2}{r^6 c^4}. \tag{2.14}$$

Indeed, as this is a scalar quantity, it has the same value in any coordinate system at a given point of the manifold. As it diverges for $r \to 0$, the Schwarzschild solution harbors a true singularity at $r = 0$, with a diverging gravitational field.

The surface at $r = R_S$ is called a *horizon*. When looking a bit more in detail at it, one sees that it is a null surface: it is tangent to lightcones. It can be seen as the place where outgoing null geodesics remain stabilized by the gravitational field. Photons from inside this surface cannot escape and all timelike or null geodesics in this region end at the central singularity. In particular, this means that no signal can be sent from inside the black hole to the outer world. The inside is called a *black hole* and the horizon is considered as the "surface" of the black hole although there is no matter present (remember that the Schwarzschild solution is a solution of Einstein equations in **vacuum**). It is a conjecture that the collapse of any "realistic" type of matter ending in a singularity should be surrounded by a horizon (*cosmic censorship*, by Penrose), and therefore no singularity can communicate with the exterior (no *naked singularity*).

Note that there are other types of black hole solutions, which are rotating (and are eventually charged), namely the Kerr solution, but these shall not be presented here.

[2] the curvature scalar R is null in this case, see Sec. 2.1.2

2.2 Stars and tests of General Relativity

2.2.1 Tolman-Oppenheimer-Volkoff system

Let us now consider the case of a spacetime with a perfect fluid, for which the stress-energy tensor is given by Eq. (1.49), representing a spherically symmetric and static star, located for $r < R_*$. In this case let us re-write the most general metric (2.3) to the form

$$ds^2 = -e^{2\nu(r)}c^2\,dt^2 + \frac{1}{1-\frac{2Gm(r)}{rc^2}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right), \tag{2.15}$$

where the two unknown functions are now $\nu(r)$ and $m(r)$ (not depending on t, spacetime is static).

The presence of the four Killing fields (see Sec. 2.1.1) implies that the 4-velocity

$$u^\mu = \frac{u^0}{c}\partial_t{}^\mu,$$

and the norm being $u^\mu\,u_\mu = -1$, it allows us to compute

$$u^0 = e^{-\nu}. \tag{2.16}$$

The components of the stress-energy tensor (1.49) can be written

$$\begin{aligned} T_{tt} &= \varepsilon\,e^{2\nu}, \\ T_{rr} &= \frac{p}{1-\frac{2Gm}{rc^2}}, \\ T_{\theta\theta} &= p\,r^2, \\ T_{\varphi\varphi} &= p\,r^2\sin^2\theta, \end{aligned} \tag{2.17}$$

where p is the pressure, ε the energy density and the other components are zero. With the expressions for the Ricci tensor (2.5), adapted to the metric (2.15), the Einstein equations (where the Ricci scalar R is not null this time) take the form:

$$\begin{aligned} \frac{dm}{dr} &= 4\pi r^2\frac{\varepsilon(r)}{c^2}, \\ \frac{d\nu}{dr} &= \left(1-\frac{2Gm(r)}{rc^2}\right)^{-1}\left(\frac{Gm(r)c^2}{r^2} + 4\pi Gp(r)\right), \end{aligned} \tag{2.18}$$

and the conservation of the stress-energy tensor (1.48) gives the hydrostatic equilibrium:

$$\frac{dp}{dr} = -\left[\varepsilon(r) + p(r)\right]\frac{d\nu}{dr}. \tag{2.19}$$

This system of three first-order ordinary differential equations (2.18)-(2.19) is called the *Tolman-Oppenheimer-Volkoff system* (TOV), for the unknown functions $m(r), \nu(r), \varepsilon(r)$ and $p(r)$. It must be completed by a cold equation of state (no temperature dependence):

$$p = p(\varepsilon), \tag{2.20}$$

and initial (or boundary) conditions. They are the following:

- from regularity conditions at the origin $m(0) = 0$,
- the integration constant for ν is determined at the end of the integration by matching to the Schwarzschild solution (2.8) at the surface of the star (vacuum) by $g_{tt} \times g_{rr}(r = R_*) = 1$,
- $\varepsilon(0) = \varepsilon_0$, which is a parameter of the model that fixes the mass of the star.

The coordinate radius R_* is determined as the point where $p = 0$. The gravitational mass M of the star can **in this case** be simply determined by the matching to the Schwarzschild solution: it is the constant M appearing in Eq. (2.8). This mass possesses in general a maximal value, which is a typical relativistic effect: more matter produces more gravitational field, which needs more pressure to compensate for an equilibrium. Contrary to the Newtonian theory, pressure enters the sources of the gravitational field equations (2.18) so that, as some point, equilibrium is no longer possible.

2.2.2 Some experimental tests of general relativity

Until this section, the theory of general relativity has mostly been shown here as a mathematical construction. Nevertheless, there have been many experimental tests of the theory, and some of them shall be briefly described hereafter.

Gravitational redshift

The aim of this experiment is to verify that a photon emitted inside a gravitational potential, is detected at higher potential with a redshifted frequency. In 1960 Pound & Rebka compared the frequencies of a disintegration line of iron (^{57}Fe) in gamma-rays ($\lambda = 0.09$ nm), as measured at the bottom or on the top of a tower of 22m height. The redshift to be detected was of the order $z \sim 10^{-15}$, and it was confirmed with error bar of about 10%. Other emission lines have been observed to be "gravitationally redshifted" on the Sun or at the surface of white dwarves (by Greenstein and collaborators in 1971). In 1976, the space mission *Gravity Probe A* compared an atomic clock was sent into orbit and its signal compared to that of its copy on Earth. The expected redshift was much higher ($z \sim 10^{-10}$) and the agreement with general relativity was of 7×10^{-5}.

Perihelion shift of Mercury

This perihelion shift has been observed in the 19$^{\text{th}}$ century with a value of 43″ per century. This was a residual redshift after all the Newtonian corrections to the simple $1/r$ potential have been made. Indeed, any deviation from the $1/r$ Newtonian central potential accounts for a perihelion shift. The formula giving the periastron shift with general relativistic corrections has been given at the same as the publication of the theory of General Relativity:

$$\delta\varphi_{\text{peri.}} = 6\pi \left(\frac{GM}{c^2\, a(1 - e^2)} \right), \tag{2.21}$$

with M the mass of the central object, a the semi-major axis and e the eccentricity. The result of Eq. (2.21) is given in radians per orbit, but transformed in arc-seconds per century, it gives exactly the right number. Note that the "special relativistic" attempts to describe gravitation (see Sec. 1.1.3) are failing explain this observation.

Light deflection

One of the most famous tests was the measure of the light deflection by a massive body. If a photon has a trajectory that passes quite near the surface of the Sun, it should be deviated by general relativistic effects by

$$\alpha_{\max} = \frac{4GM_\odot}{R_\odot c^2} \simeq 1.7''. \tag{2.22}$$

This value has been checked experimentally with about 10% accuracy, observing stars close to the solar edge during a solar eclipse in 1919 by Eddington. Note that a Newtonian calculation assuming that photons have masses in relation with their energy, the formula (2.22) would be smaller by a factor 2. This shows that one must take into account the curvature predicted by general relativity. This light deflection is now broadly used in astrophysics to map the mass distribution of our Universe through *gravitational lenses*.

Let us mention here the future test we are all waiting for, and that shall be discussed in many of the lectures of this school: the direct detection of gravitational waves.

2.3 Gravitational radiation

2.3.1 Linearized Einstein equations

General relativity is a non-linear theory: the gravitational field itself is source of the gravitational field equations. This can be seen more precisely by setting

$$g_{\mu\nu}(x^\rho) = \eta_{\mu\nu} + h_{\mu\nu}(x^\rho), \tag{2.23}$$

with $\eta_{\mu\nu}$ the Minkowski metric (1.14) and $h_{\mu\nu}$ a perturbation. It is convenient to introduce the auxiliary variables

$$\begin{aligned} \bar{h}_{\mu\nu} &= h_{\mu\nu} - \frac{1}{2} h\, \eta_{\mu\nu}, \\ h &= \eta^{\mu\nu} h_{\mu\nu}. \end{aligned} \tag{2.24}$$

One can then show that Einstein equations (1.53) can be formally written as an infinite non-linear development in powers of $\bar{h}_{\mu\nu}$ and its derivatives. Separating the linear terms (in $\bar{h}_{\mu\nu}$), one can write

$$\Box \bar{h}_{\mu\nu} - \partial_\mu \bar{W}_\nu - \partial_\nu \bar{W}_\mu + \eta^{\rho\sigma} \partial_\rho \bar{W}_\sigma \eta_{\mu\nu} = -\frac{16\pi G}{c^4} \left[T_{\mu\nu} + t_{\mu\nu}\left(\bar{h}\right)\right], \tag{2.25}$$

where $\Box = \eta^{\rho\sigma}\partial_\rho\partial_\sigma = -c^{-2}\partial_t^2 + \Delta$ is the usual wave operator (or "d'Alembert" operator), and

$$\bar{W}_\mu = \eta^{\rho\sigma}\partial_\rho\bar{h}_{\mu\sigma}. \tag{2.26}$$

Note that within this section, indices shall lowered and raised using the flat metric $\eta_{\mu\nu}$. On the right-hand side of (2.25) the stress-energy tensor $T_{\mu\nu}$ has an additional contribution, which depends quadratically on $\bar{h}_{\mu\nu}$:

$$t_{\mu\nu} = \mathcal{O}\left(\bar{h}^2\right).$$

Let us stress here that $t_{\mu\nu}$ is not a tensor: from the equivalence principle, it is possible to remove the effect of any gravitational field in a locally inertial frame. Therefore, in such a frame the stress-energy of the gravitational field would be zero, and thus in any frame by the formula for the change of coordinates of a tensor (1.13).

Until now, the nothing has been said about the gauge choice. One can check that the left-hand side of Eq. (2.25) is invariant under the coordinate change

$$x'^\mu = x^\mu + \xi^\mu, \tag{2.27}$$

where ξ^μ is a given vector field. The perturbation $\bar{h}_{\mu\nu}$ transforms as

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\nu\xi_\mu - \partial_\mu\xi_\nu + \eta_{\mu\nu}\partial_\rho\xi^\rho, \tag{2.28}$$

and

$$\bar{W}'_\mu = \bar{W}_\mu - \Box\xi_\mu. \tag{2.29}$$

Therefore, one can always find a vector field such that

$$\bar{W}_\mu = \partial^\nu\bar{h}_{\mu\nu} = 0. \tag{2.30}$$

This condition (2.30) is called the *harmonic gauge*, or *Lorenz gauge* in analogy with electromagnetism. In such a gauge, the linearized Einstein equations take the form

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}. \tag{2.31}$$

One can see that $\bar{h}_{\mu\nu}$ represents a quantity that propagates as a wave at the speed of light c on a flat background: the *gravitational waves*.

2.3.2 Propagation in vacuum

We here consider the case of propagation of gravitational waves in vacuum:

$$\Box\bar{h}_{\mu\nu} = 0. \tag{2.32}$$

The harmonic gauge condition (2.30) does not fix all the degrees of freedom of the coordinate system, as any additional part to ξ^μ such that $\Box\xi^\mu = 0$ can still verify the harmonic gauge (2.30). To go further, let us perform a Fourier decomposition of the gravitational waves into monochromatic waves:

$$\bar{h}_{\mu\nu}(x) = \int d^4k\, A_{\mu\nu}(k) e^{ik_\rho x^\rho},$$

where k_ρ is the wave vector (with $k_\rho x^\rho = \eta_{\gamma\rho} k^\sigma x^\rho$), and $A_{\mu\nu}(k)$ the amplitude of each monochromatic wave. As $\bar{h}_{\mu\nu}$ is the solution of Eq. (2.32), one has

$$k^2 = \eta_{\mu\nu} k^\mu k^\nu = 0.$$

The wave vector is null, which is coherent with the property of gravitational waves to propagate at light speed c. The harmonic gauge condition (2.30) translates into

$$A_{\mu\nu} k^\mu = 0.$$

Let us now introduce a timelike 4-vector u^μ, associated for instance with an observer detecting the gravitational radiation. It is here important that $k_\mu u^\mu \neq 0$. One can then define a gauge, called *transverse traceless* (TT gauge) in which the amplitudes satisfy:

$$\begin{aligned} A_{\mu\nu} u^\mu &= 0 \text{ (transversality condition to } u^\mu), \\ A = \eta^{\mu\nu} A_{\mu\nu} &= 0 \text{ (traceless condition).} \end{aligned}$$

This TT gauge allows one to count the number of degrees of freedom , or polarization states, of a gravitational wave in vacuum. The 10 components of a symmetric matrix $A_{\mu\nu}$ fulfill the 4 conditions of the harmonic gauge, the 3 conditions of transversality (because one of the 4 conditions is redundant with the gauge condition), and the traceless condition. There are therefore two polarization states left for a gravitational wave. One can check that, in the observer reference frame (with $u^0 = 1$ and $u^i = 0$) and assuming that the wave is propagating in the z-direction, the matrix $h^{TT}_{\mu\nu}$ shall be given by ($\bar{h}_{\mu\nu} = h_{\mu\nu}$, because of traceless condition):

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(t - z/c) & h_\times(t - z/c) & 0 \\ 0 & h_\times(t - z/c) & -h_+(t - z/c) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.33}$$

where h_+ and $h_\times$ are two arbitrary functions describing the two polarization states of the gravitational wave. The two polarizations are called "+" and "×" (see hereafter Sec. 2.3.3 for an explanation).

If we call the complex quantity

$$H = h_+ - ih_\times, \tag{2.34}$$

it appears that it can be written in terms of the Weyl scalar (1.46) Ψ_4. Indeed, for outgoing waves in vacuum, the Weyl scalars reduce to

$$\begin{aligned} \Psi_0 &= \Psi_1 = \Psi_2 = \Psi_3 = 0, \qquad (2.35) \\ \Psi_4 &= -\ddot{h}_+ + i\ddot{h}_\times = -\ddot{H}. \end{aligned}$$

2.3.3 Effects of gravitational waves on matter

How can one see the passing of a gravitational wave? First, let us try to use the TT gauge to see what happens. By definition (2.23), the full metric tensor is

$$ds^2 = -c^2\,dt^2 + \left(\delta_{ij} + h_{ij}^{TT}\right)\,dx^i\,dx^j. \tag{2.36}$$

If one derives the geodesic equations to first order in h_{ij} for a test particle only subject to gravitational interaction, one gets that this particle remains at constant coordinates when the gravitational wave passes. This is a property of the TT gauge and is a pure gauge effect: physically, if one considers the distance between two such particles as measured by photons, one shall notice a change when a gravitational wave passes. In order to compute measurable distances (and not coordinate ones) one must use *e.g.* Fermi coordinates.

The *Fermi coordinates* allow for a description of the movement of point masses under the action of a gravitational wave in a quasi-Newtonian way. To do so, we shall admit that we can build in the neighborhood of the whole worldline of one such mass a local inertial frame, that deviates from the flat metric quadratically in the distance to this worldline. The difference here with usual local inertial frames (as in Sec. 1.2.4) is that this system of coordinates holds not only in the neighborhood of a point, but in the neighborhood of a whole geodesic. We consider a set of non-interacting point masses in the neighborhood of an observer following this worldline.

The line element of the metric in the Fermi coordinates $\{\hat{x}^\mu\}$ takes therefore the Minkowski form, in the vicinity of the observer:

$$ds^2 = -\left(d\hat{x}^0\right)^2 + \delta_{ij}d\hat{x}^i d\hat{x}^j + \mathcal{O}\left(\left|\hat{x}^i\right|^2\right) dx^\mu dx^\nu. \tag{2.37}$$

The transformation from TT gauge (2.36) to the Fermi one (2.37) is then given by

$$\hat{x}^0 = x^0, \tag{2.38}$$

$$\hat{x}^i = x^i + \frac{1}{2}h_{ij}^{TT}(t,0)x^j. \tag{2.39}$$

Assuming now that the gravitational wavelength is much greater than the typical size of the system of point masses, we can write the time evolution of the point masses in Fermi coordinates. The spatial TT coordinates (x_0^i) of this point mass do not change as the gravitational wave passes, we can then write from Eq.(2.38):

$$\hat{x}^i(t) = x_0^i + \frac{1}{2}h_{ij}^{TT}(t,0)\,x_0^j. \tag{2.40}$$

This formula can be applied to a monochromatic wave propagating in the z-direction (2.33):

$$\hat{x}(t) = x_0 + \frac{1}{2}\left(h_+ x_0 + h_\times y_0\right)\,e^{i\omega t}, \tag{2.41}$$

$$\hat{y}(t) = y_0 + \frac{1}{2}\left(h_\times x_0 - h_+ y_0\right)\,e^{i\omega t},$$

$$\hat{z}(t) = z_0. \tag{2.42}$$

A circle of particles shall be deformed as the gravitational wave passes, by alternative contractions and elongations along the $\hat{x}$ and $\hat{y}$ axes, for the polarization $+$, and along the lines $\hat{y} = \hat{x}$ and $\hat{y} = -\hat{x}$ for the polarization $\times$.

2.3.4 Generation of gravitational waves

We describe here the generation of gravitational waves by isolated systems, and we consider again the linearized version of Einstein equations (2.31), for weakly relativistic, slowly varying source, *i.e.* $T_{\mu\nu}$ does not change during a light crossing time of the source, with compact support. Under these hypothesis, Eq. (2.31) can be solved with standard retarded potential formula:

$$\bar{h}_{\mu\nu}(t, x^m) = \frac{4G}{c^4 r} \int T_{\mu\nu} \left(t - \frac{r}{c}, x'^l\right) d^3x'.$$

Using the conservation of the stress-energy tensor (1.48) to the linear order:

$$\eta^{\mu\nu}\partial_\mu T_{\nu\rho} = 0,$$

and after some algebra, one can write

$$\bar{h}_{ij}\left(t, x^m\right) = \frac{2G}{rc^4} \ddot{I}_{ij}\left(t - \frac{r}{c}\right), \tag{2.43}$$

where

$$I_{ij}(t) = \int_{\text{source}} \rho(t, x^m) x^i x^j d^3x, \tag{2.44}$$

is the tensor moment of inertia of the source (ρ is the rest-mass density).

In order to obtain the metric perturbation in th TT gauge, it is enough to consider the transverse-traceless part of Eq. (2.43). We first consider the quantity

$$Q_{ij}(t) = \int_{\text{source}} \rho(t, x^m) \left(x^i x^j - \frac{1}{3} x_k x^k \delta_{ij}\right) d^3x, \tag{2.45}$$

which is called the *mass-quadrupole moment* of the source, and which is more accessible because it enters into the multipolar development of Newtonian gravitational potential:

$$\Phi = -\frac{GM}{r} + \frac{3GQ_{ij}n^i n^j}{2r^3} + \dots$$

where $n^i = x^i/r$. With these definitions, the famous *quadrupole formula* is written as:

$$h_{ij}^{TT}(t, x^m) = \frac{2G}{r\,c^4}\left(P_i{}^k P_j{}^l - \frac{1}{2} P_{ij} P^{kl}\right) \ddot{Q}_{kl}\left(t - \frac{r}{c}\right), \tag{2.46}$$

where P_{ij} is the transverse projection operator:

$$P_{ij}(n^m) = \delta_{ij} - n_i n_j.$$

The object $t_{\mu\nu}$ introduced in Eq. (2.25) is not a tensor, but it nevertheless can have a physical meaning if we consider the *short wavelength approximation*. The idea is to consider the average of $t_{\mu\nu}$ over a region that covers several wavelengths but at the same time small compared to the characteristic lengths associated with the background metric. Indeed, one can always locally choose coordinates such that $t_{\mu\nu}$ vanishes, but this is not possible for a finite region of spacetime. This averaged stress-energy tensor comes from second-order terms in the development of Einstein equations in powers of $\bar{h}_{\mu\nu}$ (see Sec. 2.3.1):

$$T_{\mu\nu} = \langle t_{\mu\nu} \rangle = \frac{1}{32\pi G} \left\langle \partial_\mu \bar{h}_{\rho\sigma} \partial_\nu \bar{h}^{\rho\sigma} - \frac{1}{2} \partial_\mu \bar{h} \partial_\nu \bar{h} - 2 \partial_\rho \bar{h}^{\rho\sigma} \left(\partial_\nu \bar{h}_{\sigma\mu} + \partial_\mu \bar{h}_{\sigma\nu} \right) \right\rangle . \tag{2.47}$$

<> denotes averaging over several wavelengths, and this tensor is called the *Isaacson stress-energy tensor*. $T_{\mu\nu}$ is in fact gauge-invariant and, in the TT gauge it reduces to

$$T_{\mu\nu} = \frac{1}{32\pi G} \langle \partial_\mu h_{\rho\sigma} \partial_\nu h^{\rho\sigma} \rangle . \tag{2.48}$$

This tensor can also be expressed in terms of the complex quantity H (2.34):

$$T_{\mu\nu} = \frac{1}{16\pi G} \mathrm{Re} \langle \partial_\mu H \partial_\nu H \rangle . \tag{2.49}$$

In the case of a gravitational wave propagating along the z-axis, the flux of energy F transported by the wave is given by the T_{tz} component of the Isaacson tensor (2.48):

$$F = \frac{c^3}{16\pi G} \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle = \frac{c^3}{16\pi G} \left\langle \left| \dot{H} \right|^2 \right\rangle , \tag{2.50}$$

and numerically:

$$F \simeq 0.3 \left(\frac{f}{1\ \mathrm{kHz}} \right)^2 \left(\frac{h}{10^{-21}} \right)^2 \ \mathrm{W.m}^{-2}. \tag{2.51}$$

From this formula we see that gravitational waves as small as 10^{-21} are carrying a great amount of energy, and in analogy with the theory of elasticity, that spacetime is quite a rigid medium.

Integrating the definition of the Isaacson tensor (2.48) over a sphere and using the quadrupole formula (2.46), one gets the total gravitational luminosity of a given source:

$$L = \frac{dE}{dt} = \frac{1}{5} \frac{G}{c^5} \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle , \tag{2.52}$$

equivalently, using the complex quantity H, or the Weyl scalar:

$$\lim_{r \to +\infty} \frac{r^2 c^3}{16\pi G} \int_{\mathrm{sphere}} \left| \dot{H} \right|^2 d\Omega = \lim_{r \to +\infty} \frac{r^2 c^3}{16\pi G} \int_{\mathrm{sphere}} \left| \int_{-\infty}^{t} \Psi_4 dt' \right|^2 d\Omega. \tag{2.53}$$

Eq. (2.52) can be transformed to get an order-of-magnitude estimate for a source of mass M, size R, typical pulsation ω:

$$L \sim \frac{G}{c^5} s^2 \omega^6 M^2 R^4,$$

where s is an asymmetry factor: $s = 0$ for a spherically symmetric source.[3] Introducing the Schwarzschild radius of the source R_S (2.10), this luminosity can be given by:

$$L \sim \frac{c^5}{G} s^2 \left(\frac{R_S}{R}\right)^2 \left(\frac{v}{c}\right)^6 . \tag{2.54}$$

With this formula, it is clear that there cannot be any type of laboratory experiment that would produce sufficiently large gravitational waves, that could be detected. Good sources include compact non-spherical objects $R \sim R_S$ moving at relativistic speeds.

Similarly, it is possible to compute the flux of linear momentum in the case of a wave traveling radially from a source toward $r \to +\infty$:

$$F_i = T_{iz} = \frac{1}{16\pi G} \mathrm{Re} \left\langle \partial_i H \partial_r H \right\rangle = \frac{1}{16\pi G} n_i \left\langle \left| \dot{H} \right|^2 \right\rangle , \tag{2.55}$$

with n_i the unit radial vector in flat space. The total flux of momentum leaving the system is given by:

$$\frac{dP_i}{dt} = \lim_{r \to +\infty} \frac{r^2 c^2}{16\pi G} \int_{\text{sphere}} l_i \left| \dot{H} \right|^2 d\Omega = \lim_{r \to +\infty} \frac{r^2 c^2}{16\pi G} \int_{\text{sphere}} l_i \left| \int_{-\infty}^{t} \Psi_4 dt' \right|^2 d\Omega. \tag{2.56}$$

The case of the flux of angular momentum is more complicated, because the averaging procedure that is used to compute the Isaacson stress-energy tensor does not take into account terms going to zero as $1/r^3$, which are precisely those contributing to the flux of angular momentum. However, it is still possible to obtain the flux of angular momentum carried away by gravitational waves:

$$\frac{dJ^i}{dt} = \lim_{r \to +\infty} \frac{r^2}{32\pi G} \int_{\text{sphere}} \varepsilon^{ijk} \left(x_j \partial_k h_{lm} + 2\delta_{lj} h_{mk} \right) \partial_r h^{lm} d\Omega. \tag{2.57}$$

2.3.5 Binary pulsar test

Gravitational waves have not yet been directly observed. Still, the study of *binary pulsars* (binary systems composed of one pulsar and another compact object) have shown indirectly the existence of gravitational radiation, as predicted by general relativity. These systems are very interesting because relativistic effects play an important role in their dynamics. For example PSR 1913+16, discovered in 1974 by Hulse and Taylor (who got the Nobel Prize in 1993) shows a 4 degrees shift in the orbit periastron, to be compared to the 43″ of Mercury's perihelion shift (see Sec. 2.2.2).

[3] remember that gravitational waves are generated by the quadrupole of a source

This binary system is emitting gravitational radiation and therefore looses orbital energy, showing a slow decay of the orbital period. One can use, in first approximation, third Kepler's law and the quadrupole formula (2.46) applied to a system made of two point masses, to obtain the time rate for the change of period:

$$\left\langle \frac{dP}{dt} \right\rangle = -\frac{192\pi}{5c^5} \left(\frac{2\pi G}{P}\right)^{5/3} \frac{m_p m_c}{(m_p + m_c)^{1/3}} \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1 - e^2)^{7/2}}, \tag{2.58}$$

where e is the eccentricity of the orbit, m_p the mass of the pulsar and m_c of its companion. In the case of PSR 1913+16, one computes

$$\left\langle \frac{dP}{dt} \right\rangle = -2.4 \times 10^{-12} \tag{2.59}$$

which is in excellent agreement with the observations (with higher-order corrections, the agreement is better than 10^{-3}). It is a remarkable check of general relativity in particular since it is done in a strong-field regime and it allows to compare with predictions coming from alternative theories. It is a quantitative evidence of the existence of gravitational waves, since alternative theories usually predict more gravitational radiation.