Description of the instrument action on polarisation

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Contents

1 Description of the radiation field, the coherence matrix **1**
- 1.1 The linear polarization basis . . . 1
- 1.2 The helicity basis . . . 2

2 Description of the instrument **3**
- 2.1 The Jones matrix . . . 3
- 2.2 Note on the reference systems . . . 4
- 2.3 The Mueller matrix . . . 4

3 The various types of instruments **4**
- 3.1 Instruments generating no circular polarisation . . . 5
- 3.2 Instruments that induce circular polarisation . . . 6

4 Polarised beams as Jones or Mueller Matrices **6**
- 4.1 Getting the Jones matrix from orthogonal antenna patterns . . . 7

A Quelques relations sur les matrices de Pauli **9**

B The matrix elements of a general Mueller matrix **9**

C The most general real Jones matrix **9**

1 Description of the radiation field, the coherence matrix

1.1 The linear polarization basis

A coherent fully polarised radiation field is represented by a 2 dimensional complex vector field orthogonals to the propagation direction. In the linear polarization basis , the radiation field is projected on two linear polarization basis $|x>$, $|y>$, or equivalently $|\| >$, $|\perp>$[1]:

$$\vec{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \tag{1}$$

which depends on 3 physical parameters, plus one irrelevant phase. In general a radiation field is an incoherent superposition of waves with different polarisations, it is then more convenient to describe the radiation field statistically through its hermitian "coherence" matrix[2] $\boldsymbol{C}_L$[3]:

$$\boldsymbol{C}_L = \begin{pmatrix} < |E_x|^2 > & < E_x\, E_y^* > \\ < E_x^*\, E_y > & < |E_y|^2 > \end{pmatrix}. \tag{2}$$

[1] The "$\|$, $\perp$" notation means that the x, y directions are taken parallel and perpendicular to some direction of the observing device, for intance a polarimeter direction.

[2] For the coherence matrix and the stokes parameters, see for instance (Born and Wolf, 1980)

[3] The index L stands for a basis of Linear polarization

Because of the mean value that enter its definition, the coherence matrix depends on 4 independent physical parameters, which reduce to 3 for a fully polarised radiation. These four physical parameters can be chosen to be the Stokes parameters:

$$\boldsymbol{C}_L = \frac{1}{2}\,(I\,\mathbb{1} + Q\,\sigma_Q + U\,\sigma_1 + V\,\sigma_V). \tag{3}$$

where matrices σ_i are the usual Pauli matrices :

$$\sigma_Q^L = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma_U^L = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_V^L = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{4}$$

The reason why we choose to index them in this way should be obvious from relations (3) and (5). The Stokes parameters:

$$I =< |E_x|^2 + |E_y|^2 >,\ Q =< |E_x|^2 - |E_y|^2 >, U = 2\ < \Re(E_x\,E_y^*) >, V = -2\ < \Im(E_x\,E_y^*) >$$

can be deduced from the coherence matrix as:

$$I = \mathrm{Tr}\,\boldsymbol{C}_L,\quad Q = \mathrm{Tr}(\sigma_Q^L\,\boldsymbol{C}_L),\quad U = \mathrm{Tr}(\sigma_U^L\,\boldsymbol{C}_L),\quad V = \mathrm{Tr}(\sigma_V^L\,\boldsymbol{C}_L) \tag{5}$$

and satisfy the inequality:

$$I^2 \geq Q^2 + U^2 + V^2,$$

which expresses the fact that the polarised energy cannot exceed the total energy. It becomes an equality for a fully polarised radiation, thus reducing the number of physical parameters to 3, as expected.

1.2 The helicity basis

It is also useful to work in the helicity basis which is a basis of circlar polarization:

$$\begin{matrix} |+> \\ |-> \end{matrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{matrix} |\parallel> \\ |\perp> \end{matrix} \quad \text{or} \quad \begin{matrix} |\parallel> \\ |\perp> \end{matrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{matrix} |+> \\ |-> \end{matrix}$$

or equivalently

$$\begin{matrix} E_+ \\ E_- \end{matrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{matrix} E_\parallel \\ E_\perp \end{matrix} \quad \text{or} \quad \begin{matrix} E_\parallel \\ E_\perp \end{matrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{matrix} E_+ \\ E_- \end{matrix}\ .$$

One can write the correlation matrix in this basis $\boldsymbol{C}_H$[4]:

$$\begin{aligned} \boldsymbol{C}_H \quad \begin{pmatrix} <|E_+|^2> & <E_+\,E_-^*> \\ <E_-\,E_+^*> & <|E_-|^2> \end{pmatrix} &= \frac{1}{2}(\mathbb{1} - i\sigma_1)\boldsymbol{C}_L(\mathbb{1} + i\sigma_1) = \frac{1}{2}\left(\boldsymbol{C}_L - i[\sigma_1, \boldsymbol{C}_L] + \sigma_1\boldsymbol{C}_L\sigma_1\right). \\ &= \frac{1}{2}(I\mathbb{1} + V\sigma_3 + U\sigma_1 - Q\sigma_2), \end{aligned}$$

où l'on a utilisé les équations 27.
L'équation 5 reste valable dans la base d'hélicité:

$$I = \mathrm{Tr}\,\boldsymbol{C}_H,\quad Q = \mathrm{Tr}(\sigma_Q^H\,\boldsymbol{C}_H),\quad U = \mathrm{Tr}(\sigma_U^H\,\boldsymbol{C}_H),\quad V = \mathrm{Tr}(\sigma_V^H\,\boldsymbol{C}_H),$$

where (the index S stands for the Stokes parameters Q, U, and V):

$$\sigma_S^H = \frac{1}{\sqrt{2}}(\mathbb{1} - i\sigma_1)\sigma_S^L\frac{1}{\sqrt{2}}(\mathbb{1} + i\sigma_1),\ \text{namely: } \sigma_Q^H = -\sigma_2,\ \sigma_U^H = \sigma_1,\ \sigma_V^H = \sigma_3.$$

[4] The index H stands for Helicity basis

2 Description of the instrument

2.1 The Jones matrix

An instrument transforms linearly the 2 dimensional vector representing the incoming radiation field $\vec{E}^{in}$ into the outgoing field $\vec{E}^{out}$, also a 2 dimensional complex vector. Therefore, the action of the instrument can in general be represented by a 2×2 complex Jones matrix $\boldsymbol{J}$ (Jones, 1941a; Jones, 1941b; Jones, 1942):

$$\begin{pmatrix} E_{\parallel}^{out} \\ E_{\perp}^{out} \end{pmatrix} = \boldsymbol{J}_L \begin{pmatrix} E_{\parallel}^{in} \\ E_{\perp}^{in} \end{pmatrix} = \begin{pmatrix} J_{\parallel\,\parallel} & J_{\parallel\,\perp} \\ J_{\perp\,\parallel} & J_{\perp\,\perp} \end{pmatrix} \begin{pmatrix} E_{\parallel}^{in} \\ E_{\perp}^{in} \end{pmatrix}.$$

The Jones matrix depends on 7 parameters plus one irrelevant phase.

If the instrument can be split into successive parts, through which the radiation travels before it reaches the bolometer, then the total $\boldsymbol{J}$ matrix is the product of the $\boldsymbol{J}$ matrices of the parts. For instance if the radiation goes first through the telescope, then through a first horn, then through a polariser and finally through a second horn, the total $\boldsymbol{J}$ matrix of the instrument is:

$$\boldsymbol{J} = \boldsymbol{J}_{Horn_2}\, \boldsymbol{J}_{Polariser}\, \boldsymbol{J}_{Horn_1}\, \boldsymbol{J}_{Telescope}\ .$$

If the Stokes parameters of the incoming radiation are I_{in}, Q_{in}, U_{in} and V_{in}, a bolometer behind the instrument receives an intensity:

$$I_{bolo} = a_I\, I_{in} + a_Q\, Q_{in} + a_U\, U_{in} + a_V\, V_{in}, \tag{6}$$

where

$$\begin{array}{l} a_I = (|J_{\parallel\,\parallel}|^2 + |J_{\parallel\,\perp}|^2 + |J_{\perp\,\perp}|^2 + |J_{\perp\,\parallel}|^2)/2 \\ a_Q = (|J_{\parallel\,\parallel}|^2 - |J_{\parallel\,\perp}|^2 - |J_{\perp\,\perp}|^2 + |J_{\perp\,\parallel}|^2)/2 \\ a_U = \Re(J_{\parallel\,\parallel}\, J^*_{\parallel\,\perp} + J_{\perp\,\perp}\, J^*_{\perp\,\parallel}) \\ a_V = \Im(J_{\parallel\,\parallel}\, J^*_{\parallel\,\perp} - J_{\perp\,\perp}\, J^*_{\perp\,\parallel}) \end{array}\ . \tag{7}$$

Amplitudes a_i satisfy the inequality:

$${a_I}^2 \geq {a_Q}^2 + {a_U}^2 + {a_V}^2,$$

which again expresses the fact that the instrument does not induce a polarised energy larger than the total energy.

The Jones matrix in terms of Pauli matrices: It is sometimes convenient to write the Jones matrix in terms of the Pauli matrices:
In the linear polarization basis:

$$\boldsymbol{J}_L = a\mathbb{1} + \vec{b}.\vec{\sigma}, \tag{8}$$

where a can be taken real and $\vec{b}$ is a complex vector. To keep track of the reality properties of the Jones matrix, which are related to the circular polarisation induced by the instrument.

$$\vec{b} = \begin{pmatrix} b_1 \\ ib_2 \\ b_3 \end{pmatrix}. \tag{9}$$

Then the Jones matrix writes:

$$\boldsymbol{J}_L = \begin{pmatrix} a + b_3 & b_1 + b_2 \\ b_1 - b_2 & a - b_3 \end{pmatrix}.$$

Any non zero imaginary part in b_i means that the Jones matrix generates circular polarization from linear polarisation or no polarization.
In the helicity basis:

$$\boldsymbol{J}_H = a + b_1\sigma_1 - b_3\sigma_2 + ib_2\sigma_3 = \begin{pmatrix} a + ib_2 & b_1 + ib_3 \\ b_1 - ib_3 & a - ib_2 \end{pmatrix}.$$

2.2 Note on the reference systems

Of course, the Jones matrix depends on the "in" and "out" reference systems which are in general different. For instance the "in" reference system is some conventional "Co-Cross" reference system orthogonal to the direction of propagation of the incoming radiation, whereas the "out" reference system lies in the focal plane[5] "Co(∥), (Cross(⊥))" directions being parallel (orthogonal) to the presumed direction of the polarizer .

Transformation of the coherence matrix by the instrument: By going through the instrument, the coherence matrix $\boldsymbol{C}_{in}$ of radiation is transformed to:

$$\boldsymbol{C}_{out} = \boldsymbol{J}\,\boldsymbol{C}_{in}\,\boldsymbol{J}^{\dagger} \tag{10}$$

2.3 The Mueller matrix

The Jones matrix describes how the components of the radiation field transform as they go through the instrument. Using equation (10), one can construct the "Mueller matrix" $\boldsymbol{M}$ (Mueller, 1948) which tells us how the Stokes parameters transform:

$$\boldsymbol{S}_{out} = \begin{pmatrix} I_{out} \\ Q_{out} \\ U_{out} \\ V_{out} \end{pmatrix} = \boldsymbol{M}\boldsymbol{S}_{in} = \boldsymbol{M}\begin{pmatrix} I_{in} \\ Q_{in} \\ U_{in} \\ V_{in} \end{pmatrix} \tag{11}$$

From equations (5) and (10), it is easy to compute the elements of the Mueller matrix from the Jones matrix:

$$M_{S\,S'} = \frac{1}{2}\mathrm{tr}\left(\sigma_S\,\boldsymbol{J}\,\sigma_{S'}\,\boldsymbol{J}^{\dagger}\right), \tag{12}$$

where the σ_S matrices have been choosen in the the same basis (linear or helicity) as the Jones matrix.Then the mueller matrix is the same in both basis, because the unitary matrix $\frac{1}{\sqrt{2}}(\mathbb{1} - i\sigma_1)$ disappears from the trace. It is easily seen that matrix $M_{S\,S'}$ is real, as it should, the Stokes parameters being real parameters, and that the Mueller matrix is symmetric when the Jones matrix $\boldsymbol{J}$ is hermitian. Expressions of the general Mueller matrix in terms of the elements of the Jones matrix and in terms of its development on the Pauli matrices (equations (8) and (9)) are given in appendix B. The amplitudes a_S which appear in Eq. (6) are in fact the elements of the first row of the Mueller matrix:

$$a_I = \boldsymbol{M}_{I\,I}, \quad a_Q = \boldsymbol{M}_{I\,Q}, \quad a_U = \boldsymbol{M}_{I\,U}, \quad a_V = \boldsymbol{M}_{I\,V}, \tag{13}$$

as can be verified by comparing equation (7) with the general expression (28) of the Mueller matrix in appendix B.

When the radiation successively goes through several instruments 1, 2 ,3 ..., the total Jones matrix is the product of the corresponding Jones matrices: $\boldsymbol{J} = ...\boldsymbol{J}_3\,\boldsymbol{J}_2\,\boldsymbol{J}_1$, The coherence matrix of the outgoing field will be

$$\boldsymbol{C}_{out} = \boldsymbol{J}\,\boldsymbol{C}_{in}\,\boldsymbol{J}^{\dagger} = ...\boldsymbol{J}_3\,\boldsymbol{J}_2\,\boldsymbol{J}_1\,\boldsymbol{C}_{in}\,\boldsymbol{J}_1^{\dagger}\,\boldsymbol{J}_2^{\dagger}\,\boldsymbol{J}_3^{\dagger}... \tag{14}$$

and by repeated use of equations (10), (11) and (12), one can see that the total Mueller matrix is also the product of the Mueller matrices of the successive instruments:

$$\boldsymbol{M} = ...\boldsymbol{M}_3\,\boldsymbol{M}_2\,\boldsymbol{M}_1. \tag{15}$$

3 The various types of instruments

We shall go through the 7 parameters of the Jones (Mueller) matrices and their relations to various possibles behaviours of the instrument. One of the prominent characteristics of an instrument is its ability to circularly polarise an initially unpolarised or linearly polarised radiation. As the circular polarisation results from a phase shift between the time oscillations of the two components of the radiation field, instruments creating circular polarisation have a complex Jones matrix.

[5]By Focal plane, we mean the tangent plane to the wave front where it reaches the feed horn.

3.1 Instruments generating no circular polarisation

They have a real Jones Matrix that depend on 4 parameters and can all be constructed as products of matrices describing the following elementary actions (see the proof in Appendix C):

An imperfect polariser: with the copolar ($\|$) direction at an angle α from the X axis (3 parameter):

$$\begin{aligned}
\boldsymbol{J}_{L,polariser} &= \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\tau_\| & 0\\ 0 & \tau_\perp\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\\
&= \tfrac{1}{2}\begin{pmatrix}\tau_\| + \tau_\perp + (\tau_\| - \tau_\perp)\cos 2\alpha & (\tau_\| - \tau_\perp)\sin 2\alpha\\ (\tau_\| - \tau_\perp)\sin 2\alpha & \tau_\| + \tau_\perp - (\tau_\| - \tau_\perp)\cos 2\alpha\end{pmatrix} \qquad (16)\\
&= \tfrac{1}{2}\left[(\tau_\| + \tau_\perp)\mathbb{1} + (\tau_\| - \tau_\perp)(\cos 2\alpha\,\sigma_3 + \sin 2\alpha\,\sigma_1)\right]
\end{aligned}$$

$\tau_\|$ and $\tau_\perp$ are the transmission coefficients in the $\|$ and $\perp$ directions, they verify $1 \geq \tau_\| \geq \tau_\perp \geq 0$ and hopefully $\tau_\| \gg \tau_\perp$. In the helicity basis, it becomes:

$$\begin{aligned}
\boldsymbol{J}_{H,polariser} &= \frac{1}{2}\left[(\tau_\| + \tau_\perp)\mathbb{1} + (\tau_\| - \tau_\perp)(-\cos 2\alpha\,\sigma_2 + \sin 2\alpha\,\sigma_1)\right]\\
&= \tfrac{1}{2}\begin{pmatrix}\tau_\| + \tau_\perp & (\tau_\| - \tau_\perp)(ie^{2i\alpha})\\ (\tau_\| - \tau_\perp)(-ie^{-2i\alpha}) & \tau_\| + \tau_\perp\end{pmatrix}.
\end{aligned}$$

The corresponding Mueller matrix is:

$$\boldsymbol{M}_{polariser} = \begin{pmatrix} K & k\cos 2\alpha & k\sin 2\alpha & 0\\ k\,\cos 2\alpha & K\,\cos^2 2\alpha + q\sin^2 2\alpha & (K-q)\sin 2\alpha\cos 2\alpha & 0\\ k\sin 2\alpha & (K-q)\sin 2\alpha\cos 2\alpha & K\sin^2 2\alpha + q\,\cos^2 2\alpha & 0\\ 0 & 0 & 0 & q\end{pmatrix} \qquad (17)$$

where $K = (\tau_\|^2 + \tau_\perp^2)/2$, $k = (\tau_\|^2 - \tau_\perp^2)/2$, and $q = \tau_\|\,\tau_\perp$, with $1 \geq K \geq k \geq 0$ and $K^2 = k^2 + q^2$.

A rotation of the polarisation: by an angle β (or of the axis by an angle $-\beta$) is described by the following Jones matrix (one parameter):

$$\boldsymbol{J}_{L,rotation} = e^{-i\,\beta\,\sigma_2} = \begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix},\ \boldsymbol{J}_{H,rotation} = e^{i\,\beta\,\sigma_3} = \begin{pmatrix} e^{i\beta} & 0\\ 0 & e^{-i\beta}\end{pmatrix}, \qquad (18)$$

and the corresponding Mueller matrix is: \label{eq:} \bmat \\ \\ \\ \emat

$$\boldsymbol{M}_{rotation} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos 2\beta & -\sin 2\beta & 0\\ 0 & \sin 2\beta & \cos 2\beta & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \qquad (19)$$

A mirror transformation: (no free parameter): ,\quad \MB = \bmat \emat

$$\boldsymbol{J}_L = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \boldsymbol{J}_H = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},\quad \boldsymbol{M} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}.$$

To summarise: the most general instrument inducing no circular polarisation can be viewed as an imperfect polarimeter in some direction α followed by a device rotating, and possibly mirroring, the polarisation. If the instrument is just in front of the measuring bolometer, only sensitive to the intensity, this last polarisation rotation and/or reflection is irrelevant and the signal in the bolometer is governed by the first row of the Mueller matrix in Eq. (17):

$$I_{bolo} = K\,I_{in} + k\,\cos 2\alpha\,Q_{in} + k\,\sin 2\alpha\,U_{in} \qquad (20)$$

3.2 Instruments that induce circular polarisation

They are linked to the three remaining parameters, and can be classified according to the type of polarisation they turn into circular polarisation. The Jones matrix are complex and the Mueller matrix have non zero non diagonal elements in the fourth row and column (linked to the V Stokes parameter).

From Q polarisation: (along one of the reference axis): The Jones matrix is

$$\boldsymbol{J}_L = e^{i\mu\sigma_1} = \begin{pmatrix} \cos\mu & i\,\sin\mu \\ i\,\sin\mu & \cos\mu \end{pmatrix} = \boldsymbol{J}_H$$

with the Mueller matrix:

$$\boldsymbol{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\mu & 0 & -\sin 2\mu \\ 0 & 0 & 1 & 0 \\ 0 & \sin 2\mu & 0 & \cos 2\mu \end{pmatrix}$$

From U polarisation: (at 45° from the reference axis): The Jones matrix is

$$\boldsymbol{J}_L = e^{i\nu\sigma_3} = \begin{pmatrix} \cos\nu + i\,\sin\nu & 0 \\ 0 & \cos\nu - i\,\sin\nu \end{pmatrix},\ \boldsymbol{J}_H = e^{-i\mu\sigma_2} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}$$

with the Mueller matrix:

$$\boldsymbol{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos 2\nu & \sin 2\nu \\ 0 & 0 & -\sin 2\nu & \cos 2\nu \end{pmatrix}$$

From no polarisation: The Jones matrix is

$$\boldsymbol{J}_L = e^{\lambda\sigma_2} = \begin{pmatrix} \cosh\lambda & -i\sinh\lambda \\ i\sinh\lambda & \cosh\lambda \end{pmatrix},\ \boldsymbol{J}_H = e^{-\lambda\sigma_3} = \begin{pmatrix} e^{-\lambda} & 0 \\ 0 & e^{+\lambda} \end{pmatrix}$$

with the Mueller matrix:

$$\boldsymbol{M} = \begin{pmatrix} \cosh 2\lambda & 0 & 0 & \sinh 2\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh 2\lambda & 0 & 0 & \cosh 2\lambda \end{pmatrix}.$$

4 Polarised beams as Jones or Mueller Matrices

For any feed, one has to know how an incoming radiation in the direction $\vec{n}$ is mapped in the focal plane at the position of the feed. This corresponds to a Jones matrix for each direction in the sky. To define the Jones matrix, one has to give oneself reference frames, both for the incoming radiation, in the plane tangent to the celestial sphere in the direction $\vec{n}$, and in the focal plane[6], for the radiation entering the feed. There are usual conventions for this, some of which are described in Ludwig (Ludwig, 1973) (see figure 1 for the Ludwig III convention). In the particular case of antennas, one only needs to know the Co and Cross amplitudes, corresponding to *the exact direction of the antenna*, in other word the first line of the Jones matrix in the Co-Cross reference frame of the antenna. But, for any change in the direction of the antenna, one needs the full Jones matrix, or equivalently, the Co and Cross amplitudes for two different directions of the antenna. In addition, for bolometric observations, the antenna is replaced by a polarimeter which is never perfect and will leak some energy in the orthogonal direction to the bolometer. Therefore, the radiation pattern in the orthogonal direction is also required, although a rough knowledge will often be sufficient.

[6] See footnote 5, page 4

4.1 Getting the Jones matrix from orthogonal antenna patterns

An antenna in the direction $\vec{e}_X$, placed at some point in the focal plane and illuminating the telescope produces a radiation field propagating in the direction $\vec{n}$ in the sky:

$$\vec{E}^X(\vec{n}) = E^X\,(A^X_{\|}(\vec{n})\,\vec{e}^{\,X}_{\|}(\vec{n}) + A^X_{\perp}(\vec{n})\,\vec{e}^{\,X}_{\perp}(\vec{n})).$$

where E^X is the field amplitude produced by the antenna, while $\vec{e}^{\,X}_{\|}(\vec{n})$ and $\vec{e}^{\,X}_{\perp}(\vec{n})$ are conventional basis vectors forming a right handed orthogonal reference frame with the direction of propagation $\vec{n}$. The only property of these vectors that we use, is that they turn by an angle α around $\vec{n}$ if the antenna, and therefore $\vec{e}_X$, is rotated by this angle α in the focal plane. The Ludwig III convention is an example satisfying this constraint. If the antenna is rotated by an angle of $\pi/2$ toward the direction Y, the basis vector in the far field reference frame are rotated in the same way:

$$\vec{e}^{\,Y}_{\|}(\vec{n}) = \vec{e}^{\,X}_{\perp}(\vec{n}), \text{and } \vec{e}^{\,Y}_{\perp}(\vec{n}) = -\vec{e}^{\,X}_{\|}(\vec{n})$$

the field produced in the same direction $\vec{n}$ in the sky becomes:

$$\vec{E}^Y(\vec{n}) = E^Y\,(A^Y_{\|}(\vec{n})\,\vec{e}^{\,Y}_{\|}(\vec{n}) + A^Y_{\perp}(\vec{n})\,\vec{e}^{\,Y}_{\perp}(\vec{n})) = E^Y\,(A^Y_{\|}(\vec{n})\,\vec{e}^{\,X}_{\perp}(\vec{n}) - A^Y_{\perp}(\vec{n})\,\vec{e}^{\,X}_{\|}(\vec{n}))$$

In other words an emitter in the focal plane illuminating the telescope with a field (E_X, E_Y) , produces in the direction $\vec{n}$ a radiation field $(E_{\|}(\vec{n}), E_{\perp}(\vec{n}))$ given by:

$$\left(E_{\|}(\vec{n}) E_{\perp}(\vec{n})\right) = \left(A^X_{\|}(\vec{n}) - A^Y_{\perp}(\vec{n}) A^X_{\perp}(\vec{n}) A^Y_{\|}(\vec{n})\right)\,\left(E^X E^Y\right),$$

where the $(X,\,Y)$ reference frame in the focal plane is defined by the two orthogonal directions of the antenna.

The principle of reciprocity tells us that an incoming radiation field $(E_{\|}(\vec{n}), E_{\perp}(\vec{n}))$ in the direction $\vec{n}$, produces in the focal plane a field

$$\left(E^X(\vec{n}) E^Y(\vec{n})\right) = \left(A^X_{\|}(\vec{n}) A^X_{\perp}(\vec{n}) - A^Y_{\perp}(\vec{n}) A^Y_{\|}(\vec{n})\right)\,\left(E_{\|}(\vec{n}) E_{\perp}(\vec{n})\right) \tag{21}$$

Therefore, by illuminating the telescope in the reverse way with two orthogonal dipoles in the focal plane, and studying their antenna patterns, one is able to evaluate the full polarised lobe as a Jones matrix depending on the direction of an incoming radiation.

$$\boldsymbol{J}_{telescope}(\vec{n}) = \left(A^X_{\|}(\vec{n}) A^X_{\perp}(\vec{n}) - A^Y_{\perp}(\vec{n}) A^Y_{\|}(\vec{n})\right) \tag{22}$$

Note that with the matrix in Eq. (21), one is able to compute the Jones matrix in any other reference frame (X', Y'), rotated by an angle α from (X, Y) in the focal plane, provided the Co-Cross reference frame chosen for the incoming radiation also rotates by the same angle α:

$$\begin{gathered}\boldsymbol{J}'_{telescope}(\vec{n}) = \begin{pmatrix} A^{X'}_{\|}(\vec{n}) & A^{X'}_{\perp}(\vec{n}) \\ -A^{Y'}_{\perp}(\vec{n}) & A^{Y'}_{\|}(\vec{n}) \end{pmatrix} \\ = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} A^X_{\|}(\vec{n}) & A^X_{\perp}(\vec{n}) \\ -A^X_{\|}(\vec{n}) & A^Y_{\|}(\vec{n}) \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \\ \tfrac{1}{2}\left((A^X_{\|} + A^Y_{\|}) + (A^X_{\|} - A^Y_{\perp})\cos 2\alpha + (A^X_{\perp} - A^Y_{\|})\sin 2\alpha\right)\end{gathered} \tag{23}$$

The first and second lines of this matrix give the Co and Cross amplitudes relative to the X' and Y' directions respectively.

The action of a polarimeter placed behind, at an angle α from the X axis, is obtained as the product $\boldsymbol{J}_{polarimeter} JB_{telescope}$, where $\boldsymbol{J}_{polarimeter}$ is given by the expression (16). If, as argued by Fosalba (Fosalba, 2000), the copolar amplitude depend only very weakly on the direction of the dipole in the focal plane, and the crosspolar amplitudes are small, then the Jones matrix is approximately:

$$\boldsymbol{J}_{telescope}(\vec{n}) = \left(A_{\|}(\vec{n})\epsilon(\vec{n}) - \eta(\vec{n}) A_{\|}(\vec{n})\right) \tag{24}$$

When working with bolometers, Mueller matrices are more illuminating, as they give directly the transformation of the Stokes parameters. The Mueller matrix associated to the Jones matrix (24) is, to first order in ϵ and η and assuming that the induced circular polarisation is negligible (The Jones matrix is nearly real):

$$\boldsymbol{M}_{telescope}(\vec{n}) = \left(|A|^2 0 A\,(\epsilon-\eta) 0 0 |A|^2 A\,(\epsilon+\eta) 0 A\,(\epsilon-\eta) - A\,(\epsilon+\eta) |A|^2 0 0 0 0 |A|^2\right)(\vec{n}) \tag{25}$$

Multiplying by the Mueller matrix of the polarimeter (Eq. **??**), one finds the coefficients:

$$\begin{array}{rcl} a_I &=& K\,|A|^2 \\ a_Q &=& k(|A|^2\cos 2\alpha - (\epsilon+\eta)\sin 2\alpha) \\ a_U &=& k(|A|^2\sin 2\alpha + (\epsilon+\eta)\cos 2\alpha) + K\,A(\epsilon-\eta) \end{array} \tag{26}$$

where $K = (\tau_\parallel^2 + \tau_\perp^2)/2$ and $k = (\tau_\parallel^2 - \tau_\perp^2)/2$

If η and ϵ are zero, then one is back to equation (20), up to a factor $|A|^2$ which is the transmission of the telescope. The whole instrument behaves as a polarimeter in the direction α in the focal plane. If η and ϵ are small but not zero, the angle α is changed to $\alpha+(\epsilon+\eta)/2$ and a_U gets a small contribution, proportional to $K\,A(\epsilon-\eta)$.

Still, the result of Fosalba (Fosalba, 2000) should be checked for all feed positions and is probably a bad approximation for far side-lobes. Therefore we think that the polarised beams as complete Jones matrices (four complex amplitudes up to a phase) should be evaluated for each feed, thus allowing to play with the polariser directions and to choose the optimal ones.

Appendices

A Quelques relations sur les matrices de Pauli

$$
\begin{aligned}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \\
(\vec a\cdot\vec\sigma)(\vec b\cdot\vec\sigma) &= (\vec a\cdot\vec b) + i\,(\vec a\times\vec b)\cdot\vec\sigma \\
(\vec a\cdot\vec\sigma)(\vec b\cdot\vec\sigma)(\vec a\cdot\vec\sigma) &= 2(\vec a\cdot\vec b)(\vec a\cdot\vec\sigma) - |\vec a|^2(\vec b\cdot\vec\sigma) \\
(\vec a\cdot\vec\sigma)(\vec b\cdot\vec\sigma)(\vec c\cdot\vec\sigma) &= (\vec a\cdot\vec b)(\vec c\cdot\vec\sigma) - (\vec a\cdot\vec c)(\vec b\cdot\vec\sigma) + (b\cdot\vec c)(\vec a\cdot\vec\sigma) + i(\vec a\cdot(\vec b\times\vec c)) \\
\vec a\times(\vec b\times\vec c) &= (\vec a\cdot\vec b)\vec c - (\vec a\cdot\vec c)\vec b
\end{aligned}
\tag{27}
$$

B The matrix elements of a general Mueller matrix

Using Eq. (12), it easy to compute the Mueller matrix in term of the matrix elements of the Jones matrix:

$$
\boldsymbol{M} = \frac{1}{2}
\begin{pmatrix}
|J_{\parallel\,\parallel}|^2 + |J_{\parallel\,\perp}|^2 + |J_{\perp\,\parallel}|^2 + |J_{\perp\,\perp}|^2 & |J_{\parallel\,\parallel}|^2 - |J_{\parallel\,\perp}|^2 + |J_{\perp\,\parallel}|^2 - |J_{\perp\,\perp}|^2 & 2\Re(J_{\parallel\,\parallel}\,J^*_{\parallel\,\perp} + J_{\perp\,\perp}\,J^*_{\perp\,\parallel}) & 2\Im(J_{\parallel\,\parallel}\,J^*_{\parallel\,\perp} - J_{\perp\,\perp}\,J^*_{\perp\,\parallel}) \\
|J_{\parallel\,\parallel}|^2 + |J_{\parallel\,\perp}|^2 - |J_{\perp\,\parallel}|^2 - |J_{\perp\,\perp}|^2 & |J_{\parallel\,\parallel}|^2 - |J_{\parallel\,\perp}|^2 - |J_{\perp\,\parallel}|^2 + |J_{\perp\,\perp}|^2 & 2\Re(J_{\parallel\,\parallel}\,J^*_{\parallel\,\perp} - J_{\perp\,\perp}\,J^*_{\perp\,\parallel}) & 2\Im(J_{\parallel\,\parallel}\,J^*_{\parallel\,\perp} + J_{\perp\,\perp}\,J^*_{\perp\,\parallel}) \\
2\Re(J_{\parallel\,\parallel}\,J^*_{\perp\,\parallel} + J_{\perp\,\perp}\,J^*_{\parallel\,\perp}) & 2\Re(J_{\parallel\,\parallel}\,J^*_{\perp\,\parallel} - J_{\perp\,\perp}\,J^*_{\parallel\,\perp}) & 2\Re(J_{\parallel\,\parallel}\,J^*_{\perp\,\perp} + J_{\parallel\,\perp}\,J^*_{\perp\,\parallel}) & 2\Im(J_{\parallel\,\parallel}\,J^*_{\perp\,\perp} - J_{\parallel\,\perp}\,J^*_{\perp\,\parallel}) \\
-2\Im(J_{\parallel\,\parallel}\,J^*_{\perp\,\parallel} - J_{\perp\,\perp}\,J^*_{\parallel\,\perp}) & -2\Im(J_{\parallel\,\parallel}\,J^*_{\perp\,\parallel} + J_{\perp\,\perp}\,J^*_{\parallel\,\perp}) & -2\Im(J_{\parallel\,\parallel}\,J^*_{\perp\,\perp} + J_{\parallel\,\perp}\,J^*_{\perp\,\parallel}) & 2\Re(J_{\parallel\,\parallel}\,J^*_{\perp\,\perp} - J_{\parallel\,\perp}\,J^*_{\perp\,\parallel})
\end{pmatrix}
\tag{28}
$$

It can also be useful to write the Mueller matrix in terms of the parameters a and $\vec b$ of the development of the Jones matrix on the Pauli matrices (Eq. (8) and (9)) :

$$
\boldsymbol{M} =
\begin{pmatrix}
|a|^2 + |b_Q|^2 + |b_U|^2 + |b_V|^2 & 2\,\Re(a\,b_Q^* + b_U\,b_V^*) & 2\,\Re(a\,b_U^* - b_V^*\,b_Q) & 2\,\Im(a\,b_V^* - b_Q\,b_U^*) \\
2\,\Re(a\,b_Q^* - b_U\,b_V^*) & |a|^2 + |b_Q|^2 - |b_U|^2 - |b_V|^2 & 2\,\Re(-a\,b_V^* + b_Q\,b_U^*) & 2\,\Im(a^*\,b_U + b_Q\,b_V^*) \\
2\,\Re(a\,b_U^* + b_V\,b_Q^*) & 2\,\Re(a\,b_V^* + b_Q\,b_U^*) & |a|^2 - |b_Q|^2 + |b_U|^2 - |b_V|^2 & 2\,\Im(a\,b_Q^* + b_U\,b_V^*) \\
2\,\Im(a\,b_V^* + b_Q\,b_U^*) & 2\,\Im(a\,b_U^* + b_Q\,b_V^*) & 2\,\Im(a^*\,b_Q + b_U\,b_V^*) & |a|^2 - |b_Q|^2 - |b_U|^2 + |b_V|^2
\end{pmatrix}
\tag{29}
$$

C The most general real Jones matrix

A 2x2 real Jones matrix can always be written as:

$$
\boldsymbol{J} = \boldsymbol{O}_1\,\boldsymbol{\mathcal{J}}\,\boldsymbol{O}_2 \quad \text{where } \boldsymbol{\mathcal{J}} = \begin{pmatrix} j_1 & 0 \\ 0 & J_2 \end{pmatrix} \quad \text{with } j_1 \ge j_2 \ge 0, \tag{30}
$$

and matrices $\boldsymbol{O}_i$ are orthogonal.

Orthogonal 2×2 matrices can all be written as a rotation matrix $\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$ or as the product of a rotation matrix by a reflexion \bmat\emat $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This proves the statements of section 3.1.

Proof: Being a symmetric and positive semi-definite matrix $\boldsymbol{J}\,\boldsymbol{J}^T$ can always be diagonalised as:

$$
\boldsymbol{J}\,\boldsymbol{J}^T = \boldsymbol{O}_1\,\boldsymbol{\mathcal{J}}^2\,\boldsymbol{O}_1^{-1} \tag{31}
$$

One can look for a matrix $\boldsymbol{O}_2$ such that

$$
\boldsymbol{J} = \boldsymbol{O}_1\,\boldsymbol{\mathcal{J}}\,\boldsymbol{O}_2
$$

If $\boldsymbol{J}$ as a non zero determinant, $\boldsymbol{O}_2$ can be obtained as:

$$\boldsymbol{O}_2 = \boldsymbol{\mathcal{J}}^{-1}\,\boldsymbol{O}_1^{-1}\,\boldsymbol{J}$$

It is easy to see that $\boldsymbol{O}_2$ is orthogonal using Eq. (31) and the fact the inverse of an orthogonal matrix is obtained by transposition:

$$\boldsymbol{O}_2\,\boldsymbol{O}_2^T = \boldsymbol{\mathcal{J}}^{-1}\,\boldsymbol{O}_1^{-1}\,\boldsymbol{J}\,\boldsymbol{J}^T\,\boldsymbol{O}_1\,\boldsymbol{\mathcal{J}}^{-1} = \mathbb{1}$$

which proves Eq. (30).

If the Jones matrix as a zero determinant, as is the case for a perfect polariser, it means that there is a direction in the incoming polarisation space, indexed by a unit vector $|n>$ such that $\boldsymbol{J}|n>=0$. In the orthogonal direction $|m>$ the action of $\boldsymbol{J}$ is $\boldsymbol{J}|m>= j_1|m'>$, where $|m'>$ is a unit vector defining a direction of the outgoing polarisation space (and $|n'>$ the orthogonal one). Then $\boldsymbol{J}$ can be written as:\emat\bmat \emat,

$$\boldsymbol{J} = \begin{pmatrix} <i'|m'> & <i'|n'> \\ <j'|m'> & <j'|n'> \end{pmatrix} \begin{pmatrix} j_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} <m|i> & <m|j> \\ <n|i> & <n|j> \end{pmatrix},$$

which is Eq. (30) ($|i>$, $|j>$ and $|i'>$, $|j'>$ are orthonormal basis vectors in the incoming and outgoing polarisation spaces).

References

Born, M. and Wolf, E.: 1980, *Principles of Optics, sixth edition*, Pergamon Press

Fosalba, P.: 2000, *Work on Planck simulations*, Presentation at the 2^{nd} Planck Scan Strategy meeting, Garching

Jones, R. C.: 1941a, *J. Opt. Soc. Am.* **31**, 488

Jones, R. C.: 1941b, *J. Opt. Soc. Am.* **31**, 500

Jones, R. C.: 1942, *J. Opt. Soc. Am.* **32**, 486

Ludwig, A.: 1973, *IEEE Transactions on Antennas and Propagation* **AP-21**, 116

Mueller, H.: 1948, *J. Opt. Soc. Am.* **38**, 661

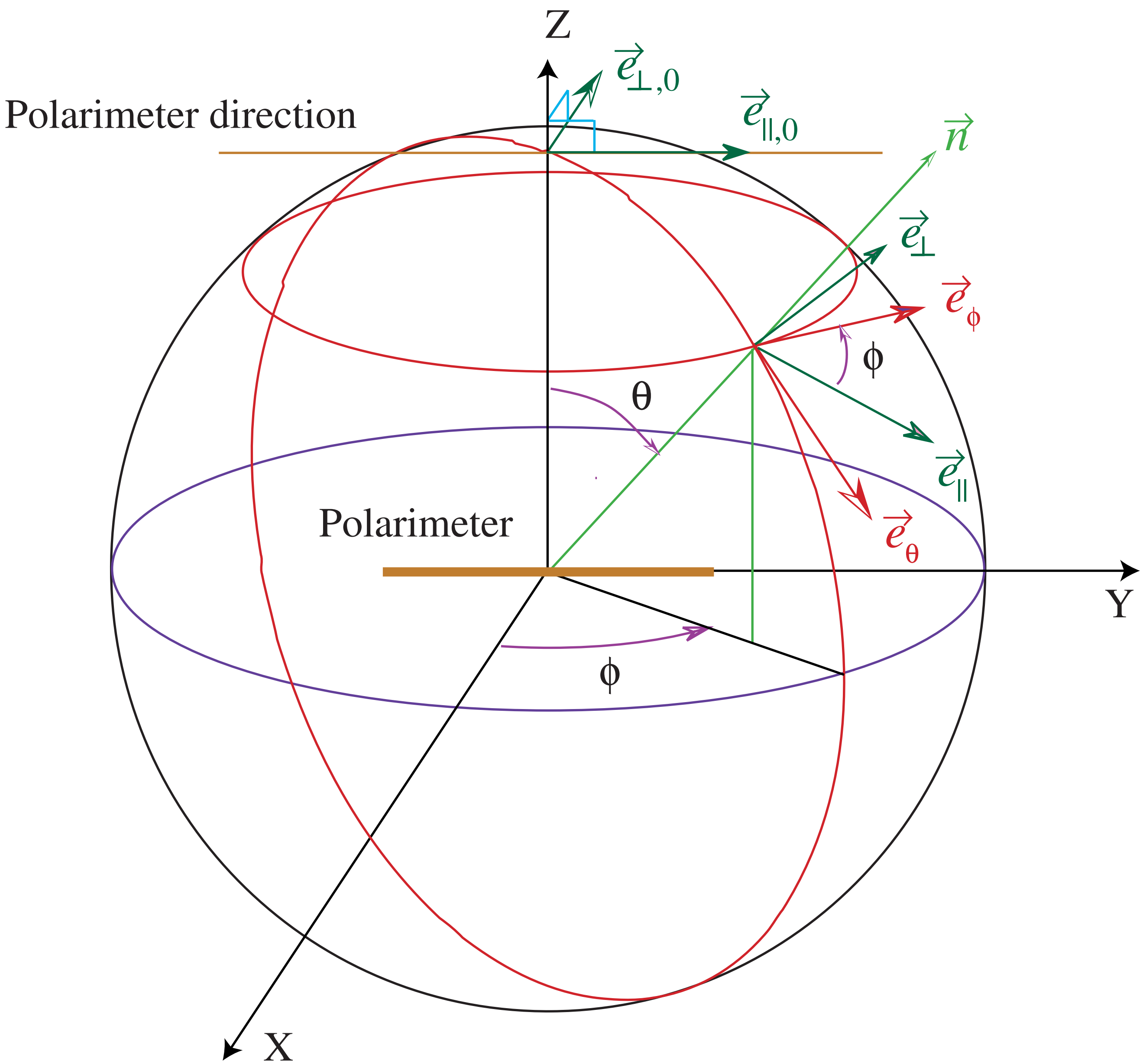

Figure 1: Co and Cross basis vector in the Ludwig III convention