On the time-frequency detection of chirps and its application to gravitational waves

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Abstract. The question of the detection of gravitational waves using time-frequency distributions is addressed. Strategies based on line integration are discussed with respect to optimality and adequacy of a representation. An effective implementation by means of reassigned spectrograms is proposed.

1 A model for gravitational waves

It is generally admitted that the most promising source of detectable gravitational waves should be produced by the coalescence of very massive binary systems which is the only situation we will consider here. In a first (Newtonian) approximation, an explicit form can be given for the expected waveform. Up to some unknown phase, it can be expressed as the real part of the complex-valued signal [11, 10]

$$x(t;t_0,d) = A\,(t_0-t)^{-\alpha}\,e^{-i2\pi d(t_0-t)^{\beta}}\,U(t_0-t), \tag{1}$$

with $\alpha = 1/4$, $\beta = 5/8$ and $U(.)$ the unit step function. In this expression, t_0 is the coalescence time and d and A are constants which mainly depend on the individual masses of the objects and, of course, on other geometrical quantities such as the distance of the binary from earth or the relative orientation between the wavefronts and the detector.

Considering (1) as a chirp, its frequency spectrum can be obtained by means of a stationary phase approximation, leading to the following result [4, 10]:

$$X_{r,k}(f) = C\,f^{-(r+1)}\,e^{i\Psi_k(f)}\,U(f), \tag{2}$$

where $\Psi_k(f) = -2\pi\left(cf^k + t_0 f + \gamma\right)$, $k = -5/3$ and $r = 1/6$. From this result, it is possible to introduce the group delay, which basically locates the time of arrival of each frequency, $t_X(f) = -1/(2\pi) d\Psi_k/df = t_0 + ckf^{k-1}$. The values of c and C depend on the individual masses of both objects of the binary through a combination referred to as "chirp mass" [11].

It has to be noted that this approximation is valid only in a frequency bandwidth which can be quantitatively determined [3] by an upper-bound of the relative approximation error written as a integral remainder [7].

2 Detection scheme

2.1 Optimal filter

Signal detection is usually considered from the point of view of the binary hypothesis testing problem (see, e.g., [13])

$$H_0 \quad : \quad r(t) = n(t) \quad ; H_1 \quad : \quad r(t) = n(t) + s(t),$$

where $s(t)$ is the reference signal to detect (supposed to be known and of finite energy over the observation time interval), $n(t)$ is some additive noise and $r(t)$ is the available observation upon which the decision has to be taken.

Given this framework, designing an "optimal" detector depends not only on the *a priori* knowledge one may have on the signal and on the noise, but also on the choice of a criterion for optimality. A relevant concept in such a search for optimality is that of "likelihood ratio test" (LRT), which essentially consists in evaluating a test statistic based on conditional probability density functions of the observation, comparing it with a threshold, and deciding that the expected signal is indeed present when the threshold is exceeded.

For a sake of simplicity, $n(t)$ will be assumed to be zero-mean, Gaussian, and stationary with a power spectral density equal to $\Gamma_n(f)$. We are here interested in the situation where the expected signal is expressed as $s(t) = x(t;\theta)\, e^{i\gamma}$, where θ is a vector of unknown parameters that we may wish to estimate and γ some unknown random phase uniformly distributed over $[0, 2\pi]$, that we would like to eliminate. In this case, the notion of LRT has to be extended to that of generalized LRT (GLRT) which, after some manipulations, simply reduces to

$$\Lambda^w(r) = \left| \int_0^{+\infty} \frac{R(f)\, \overline{X(f;\theta)}}{\Gamma_n(f)}\, df \right|^2 . \tag{3}$$

The strategy invoked here is exceedingly simple, since it only amounts to correlating the observation with a replica of the expected waveform ("matched filtering"). Because of the corrupting random phase, the optimum GLRT detector turns out, in fact, to coincide with a matched filter followed by an envelope detector, a structure referred to as "quadrature matched filtering." [13]

2.2 Time-frequency formulation

In the case where the signal to detect $X(f;\theta)$ is a chirp, a reformulation of the optimal detector (3) can be given *via* a path integration in the time-frequency plane. This results from the combination of two ingredients: (*i*) the existence of a unitary time-frequency distribution which then satisfies a Moyal-type formula such that the squared inner product (3) can be equivalently expressed in the time-frequency plane, (*ii*) the perfect localization of the corresponding time-frequency distribution along the group delay of the chirp we would like to detect.

As far as the perfect localization along the specific group delay of the chirp (2) is concerned, the only solution for the time-frequency distribution is the *so-called* Bertrand distribution [2] :

$$P_X^{(k)}(t,f) = f^{2(r+1)-q} \int_{-\infty}^{+\infty} \mu_k(u)\, X\left(f\lambda_k(u)\right) \overline{X\left(f\lambda_k(-u)\right)}\, e^{i2\pi t f \zeta_k(u)}\, du, \quad (4)$$

with $\zeta_k(u) = \lambda_k(u) - \lambda_k(-u)$, $\mu_k(u) = \dot{\zeta}_k^{1/2}(u)\, \left(\lambda_k(u)\,\lambda_k(-u)\right)^{r+1}$ and $\lambda_k(u) = \left(k(e^{-u}-1)/(e^{-ku}-1)\right)^{1/(k-1)}$. Unfortunately, this distribution (referred to as "active") is not unitary when $k \neq 0$. It is however possible to introduce a dual ("passive") distribution $\tilde{P}_X^{(k)}$ by simply changing $\mu_k(u)$ in (4), thus leading to the Moyal-type formula

$$\left| \int_0^{+\infty} X(f)\, \overline{Y(f)}\, f^{2r+1}\, df \right|^2 = \int_{-\infty}^{+\infty} \int_0^{+\infty} \tilde{P}_X^{(k)}(t,f)\, P_Y^{(k)}(t,f)\, f^{2q}\, dt\, df.$$

Together with the perfect localization property of $P_X^{(k)}$, the test statistic (3) can therefore be rewritten as

$$\Lambda^w(r; t_0, c) = C^2 \int_0^{+\infty} \tilde{P}_Z^{(k)}(t_X(f), f)\, f^{q-1}\, df. \quad (5)$$

where $Z(f) = R(f) f^{-(2r+1)}/\Gamma_n(f)$.

2.3 Simplifications

The exact formulation (5) involves a very heavy computational burden and, in order to end up with a feasible solution, it is necessary to consider simpler, yet accurate approximations. Whereas such a simplification may not be possible in the general case, it turns out that it can be effectively achieved in the specific case of gravitational waves, thanks to the specific values of the physical parameters which are involved.

From passive to active distributions. Due to low-frequency (seismic noise) and high-frequency (photon noise) limitations, the effective observation bandwidth is necessarily restricted to some bandpass frequency interval (typically $[50\text{Hz}, 500\text{Hz}]$). Within this frequency band, we can consider in a first approximation (see, e.g., [8]) that the power spectral density $\Gamma_n(f)$ of the observation noise $n(t)$ has essentially a continuous background which behaves as $\Gamma_n(f) = \sigma^2 f^{-\epsilon}$, with $\epsilon \approx 1$.

Taking this information into account, a narrow-band approximation of $\tilde{P}_Z^{(-5/3)}$ leads to a new expression for the detector (5) which only involves the active Bertrand distribution of the observation:

$$\Lambda^w(r; t_0, c) \approx \frac{C^2}{\sigma^4} \int_0^{+\infty} P_R^{(-5/3)}(t_X(f), f)\, f^{q+2\epsilon-11/3}\, df. \quad (6)$$

From active distributions to reassigned spectrograms. Given the simplified structure (6), the final problem reduces to finding some accurate and easy-to-compute approximation to the Bertrand distribution $P_R^{(-5/3)}(t, f)$. Since the key feature of this distribution is to satisfy the perfect localization on "matched" chirps, the solution that we propose is to replace it by a *reassigned spectrogram* [1] $S_X(t, f)$ which, when applied to the same power-law chirps, is known to be approximately "Dirac" along the corresponding group delay line. The effectiveness of this approximation is illustrated in [6]. The final form of the approximated optimum detector then reads

$$\Lambda^w(r; t_0, c) \approx \frac{C^2}{\sigma^4} \int_0^{+\infty} S_R\left(t_X(f), f\right)\, f^{-1/3}\, df.$$

3 Simulations and perspectives

Figure 1 presents two different examples based upon one of the typical situations discussed in [8]. Both examples assume that the binary consists of two objects of $1M_\odot$ and $10M_\odot$ (coalescence time set to $t_0 = 0$). The binary is located at and distance of 200 Mpc from earth in the first example (Fig. 1a) 1 Gpc in the second one (Fig. 1b). Since the distance between the binary and earth changes only the signal amplitude, the signal to noise ratio (SNR) is the only parameter which has been modified between these two examples. The simulation was run by corrupting the data with Gaussian additive noise, with $\epsilon = 1$ and $\sigma^2 = 0.7 \times 10^{-42}/\mathrm{Hz}$ over a frequency range of $[50\mathrm{Hz}, 500\mathrm{Hz}]$. The proposed strategy, based on a reassigned spectrogram, does not reach the ideal performance predicted by the matched filter theory, because of the limited accuracy of the different approximations which have been involved in its derivation. However, the figure evidences that it clearly allows for the detection of the chirp and that it also over-performs a crude path integration based on a standard spectrogram. Beyond detection, an estimation of the chirp mass can be achieved when this parameter is unknown (see Fig. 2) by applying systematically the same strategy over a number of integration paths corresponding to different chirp masses. The complexity of this approach can be reduced by introducing a hierarchical strategy based on time-frequency ribbons (see Fig. 3). From another perspective, replacing integration curves by ribbons also offers a way of making the detector more robust in cases where the used reference would be slightly mismatched to the actual one (e.g., Newtonian vs. post-Newtonian approximation).

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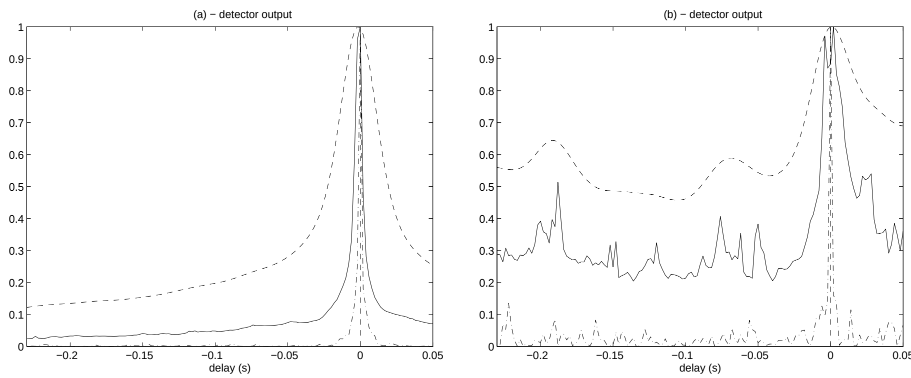

Figure 1: **Comparison between three different detectors with two different SNR's** (see Sect. 3): squared envelope of the output of the matched filter (dashed-dotted line), time-frequency strategy based on a line integration over either a classical spectrogram (dashed line) or its reassigned version (solid line). In order to make appear what is gained in terms of contrast, the maximum of each curve has been arbitrarily normalized to unity.

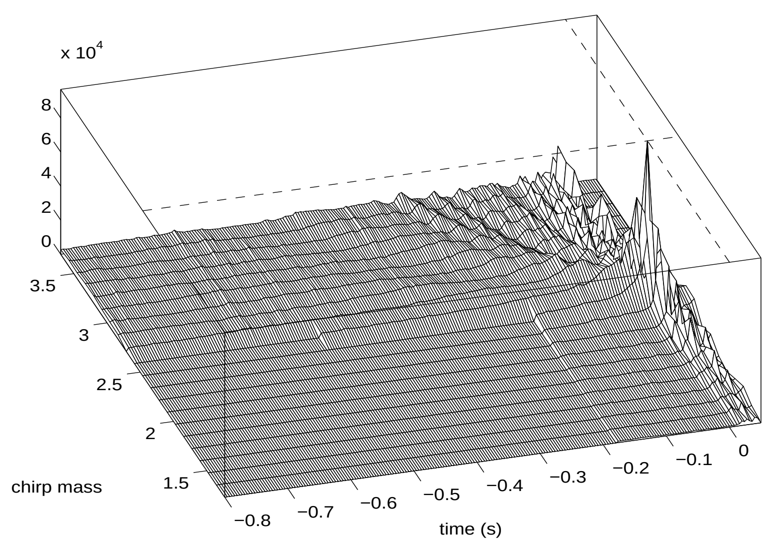

Figure 2: **Joint detection-estimation for gravitational waves**. In the case where the chirp mass $\mathcal{M}_{\odot}$ is unknown, different line integrations (similar to that of Fig. 1, but over a number of different time-frequency curves) have to be performed, here on the reassigned spectrogram. This results in a surface whose maximum allows for the detection of the gravitational wave (when it exceeds some prescribed threshold) and for the estimation of both the time of coalescence and the chirp mass (actual values are indicated with dashed lines).

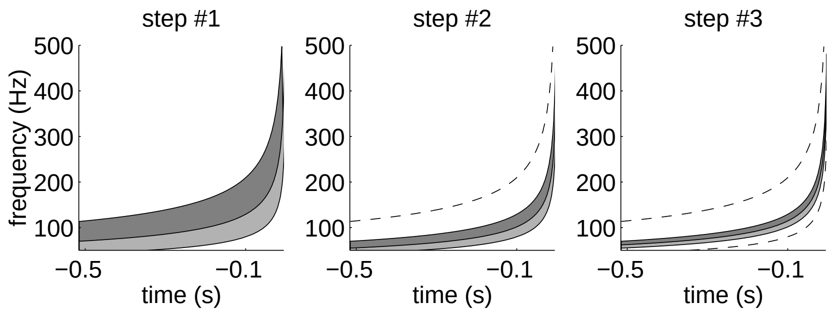

Figure 3: **Hierarchical time-frequency detection strategy**. The idea is to replace a direct systematic search among a set of chirp masses by an iterative procedure in order to reduce the computational cost. At each step, the integration is performed over two adjacent time-frequency ribbons whose supports are determined by given chirp mass intervals. A maximum energy criterion is then used for selecting the ribbon over which the procedure is iterated.

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