Gravitational waves energy spectrum of hyperbolic encounters

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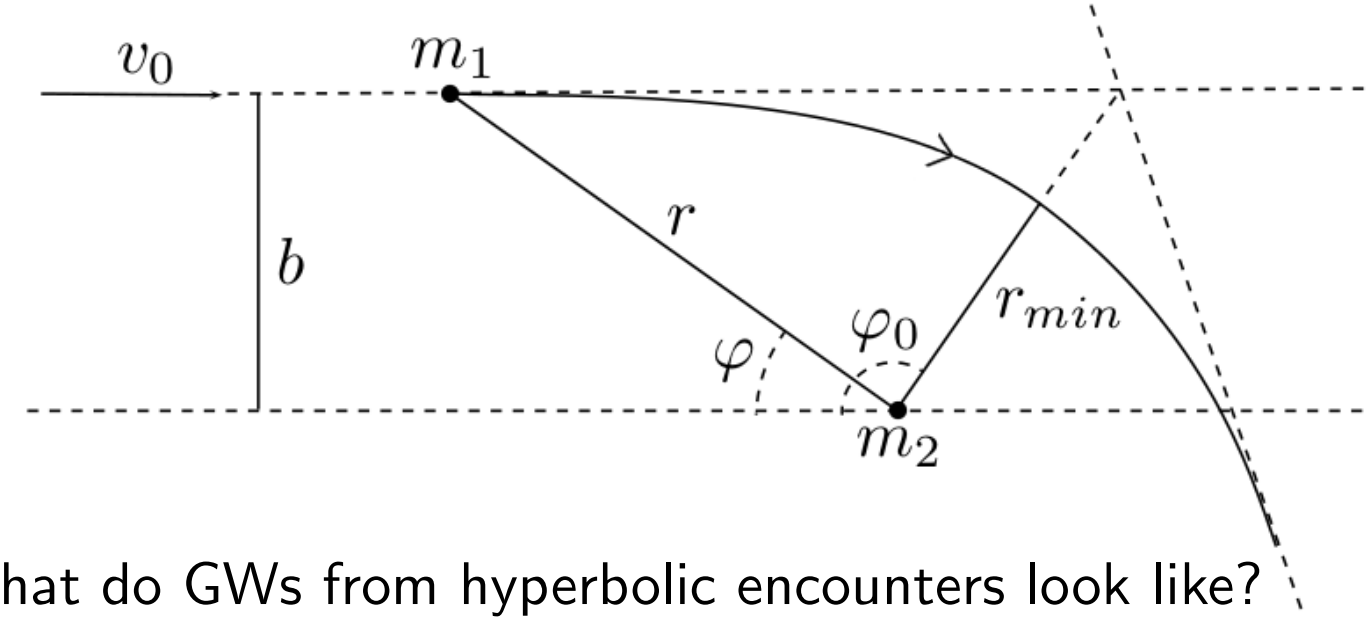
9th LISA Symposium, Paris, May 2012

Hyperbolic encounters and GWs



- What do GWs from hyperbolic encounters look like?
- How much energy and power do they carry out of the system?
- Which frequencies do they have? Where are the peaks?
- Can they be detected and useful for the astrophysics of eLISA?

Already-known quantities - I

Peters & Mathews: the energy spectrum for keplerian orbits:

$$P(n) = \frac{32\, G^4\, M_1^2\, M_2^2\, (M_1 + M_2)(1-\varepsilon)^5}{5\; c^5\; r_p^5} \cdot g(n, \varepsilon) \qquad (\, J_\nu(z)\,)$$

Berry & Gair: the energy spectrum for the parabolic limit $\varepsilon \to 1$:

$$P(f) = \frac{4\,\pi^2\, G^3\, M_1^2\, M_2^2}{5\; c^5\; r_p^2} \cdot \ell(f) \qquad (\, K_\nu(z)\,)$$

Turner: the energy spectrum for the limit $\varepsilon \to \infty$:

$$P(\sigma) = \frac{G^{7/2} m^{1/2}\, m_1^2\, m_2^2}{c^5 \quad r_p^{3.5}} \frac{8}{15\pi} \sqrt{\varepsilon} \cdot \mathcal{T}(\sigma) \qquad (\, K_\nu(z)\,)$$

Already-known quantities - II

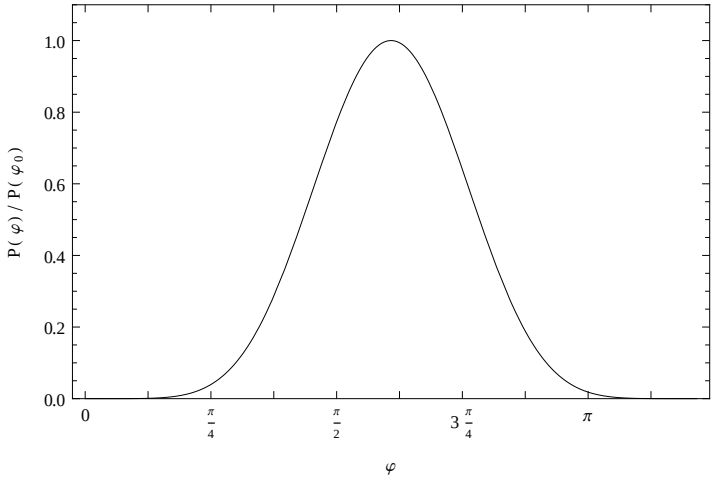
Capozziello et al.: total released energy from hyperbolic paths:

$$E = \frac{32\, G\, \mu^2\, v_0^5}{b\, c^5} \cdot F(\varphi_0) \qquad (\varepsilon)$$

Turner: total released energy from hyperbolic paths:

$$E = \frac{8\, G^{7/2}\, m^{1/2}\, m_1^2\, m_2^2}{15\, c^5\, r_p^{3.5}} \cdot g(\varepsilon) \qquad (\varepsilon)$$

Energies are computed integrating the power emitted per unit angle



Deriving the formula

Starting from the quadrupole formula $P = \gamma \langle \dddot{D}_{ij}(t)\, \dddot{D}_{ij}(t) \rangle$, we make use of Parseval's theorem - $\int |f(t)|^2 \mathrm{d}t = \int |\mathcal{F}(\omega)|^2 \mathrm{d}\omega$:

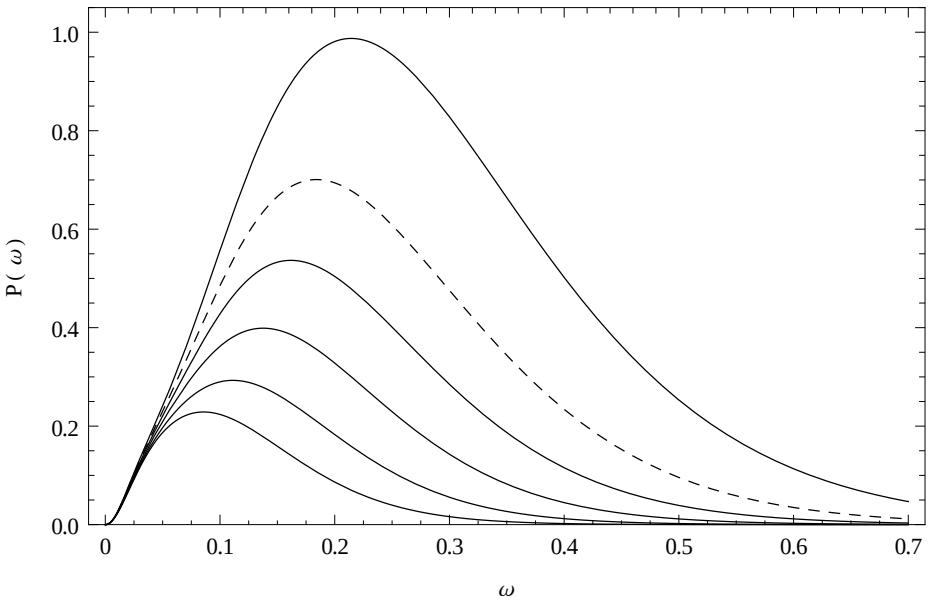
$$\Delta E = \int P(t)\mathrm{d}t = \int \gamma \langle\, \dddot{D}_{ij}(t)\, \dddot{D}_{ij}(t)\, \rangle \mathrm{d}t = \int \underbrace{\gamma \langle\, \dddot{D}_{ij}(\omega)\, \dddot{D}_{ij}(\omega)\, \rangle}_{P(\omega)} \mathrm{d}\omega$$

Compute $D_{ij}(t)$, the Fourier transform $\dddot{D}_{ij}(\omega)$, and $\langle \dddot{D}_{ij}(\omega)\, \dddot{D}_{ij}(\omega) \rangle$

For the Fourier transform of $D_{ij}(t)$ we used the procedure of Landau & Lifshitz for the energy spectrum analysis of a similar setting in electrodynamics.

The formula for the energy spectrum

$$P(\omega) = -\frac{G\, a^4\, m^2\, \pi^2}{720\, c^5}\, \omega^4\, F_\varepsilon(\omega)$$



where the function $F_\varepsilon(\omega)$ is:

$$|[16\,\varepsilon\, H^{(1)\prime}_{i\nu}(i\nu\varepsilon) + (\varepsilon^2 - 3)\, H^{(1)\prime}_{i\nu}(i\nu\varepsilon/2)]|^2 +$$
$$|[(3 - 2\,\varepsilon^2)\, H^{(1)\prime}_{i\nu}(i\nu\varepsilon/2) - 8\,\varepsilon\, H^{(1)\prime}_{i\nu}(i\nu\varepsilon)]|^2 +$$
$$|[8\,\varepsilon\, H^{(1)\prime}_{i\nu}(i\nu\varepsilon) + \varepsilon^2\, H^{(1)\prime}_{i\nu}(i\nu\varepsilon/2)]|^2 +$$
$$\frac{9\,(\varepsilon^2 - 1)}{\varepsilon^2}\, |[H^{(1)}_{i\nu}(i\nu\varepsilon/2) - 4\,\varepsilon\, H^{(1)}_{i\nu}(i\nu\varepsilon)]|^2$$

Hankel functions

$$H^{(1)}_\alpha(z) = J_\alpha(z) + \imath\, Y_\alpha(z)$$

Numerical integration

We checked numerically that $\int P(\omega)\mathrm{d}\omega = \Delta E$, i.e. that

$$\int_0^\infty \underbrace{\frac{G\, a^4\, m^2\, \pi^2}{720\, c^5}\, \omega^4\, F_\varepsilon(\omega)}_{\text{our energy spectrum}}\, \mathrm{d}\omega \quad = \quad \underbrace{\frac{32\, G\, \mu^2\, v_0^5}{b\, c^5} \cdot F(\varphi_0)}_{\text{energy emitted as found by Capozziello et al.}}$$

$$= \quad \underbrace{\frac{8\, G^{7/2}\, m^{1/2}\, m_1^2\, m_2^2}{15\, c^5\, r_p^{7/2}} \cdot g(\varepsilon)}_{\text{energy emitted as found by Turner}}$$

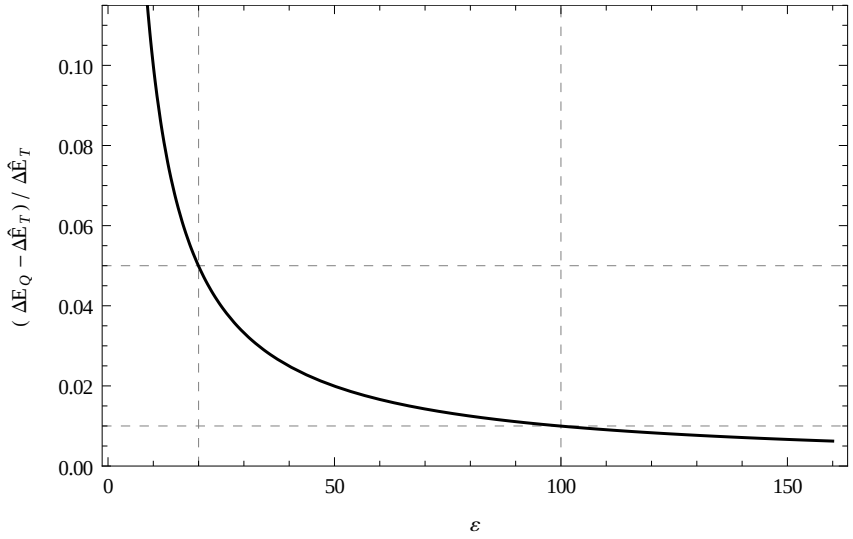
Since we found an analytical identity between these two ΔE .

The limit of large ε

Turner uses some approximations for $g(\varepsilon)$ and for the trajectories in the limit of $\varepsilon \gg 1$. Thus he can compute the power spectrum

$$P(\sigma) = \frac{G^{7/2} m^{1/2}\, m_1^2\, m_2^2}{c^5 \quad r_p^{3.5}} \frac{8}{15\pi} \sqrt{\varepsilon} \cdot \mathcal{T}(\sigma) \qquad \text{where } \sigma = \omega\, \tau$$

In that limit our results coincide



The limit of $\varepsilon = 1$

According to Berry & Gair: $P(f) = \frac{4\,\pi^2\,G^3\,M_1^2\,M_2^2}{5\,c^5\,r_p^2} \cdot \ell(f)$.

$\ell(f)$ is in terms of modified Bessel functions of the second kind, i.e. $K_\nu(z)$, and in our result $F_\varepsilon(\omega)$ is instead in terms of Hankel functions, i.e. $H_\nu(z)$, also a combination of Bessel functions.

We compared the two results numerically and found that they do agree taking the parabolic limit of $\varepsilon \to 1$.

Conclusions

- We found an exact expression for the energy spectrum in the case of hyperbolic encounters
- The formula is in agreement with the previous known results in the limits of $\varepsilon \to 1$ and $\varepsilon \to \infty$, as well as for $E = \int P$
- Our results could be useful for a rough estimate of the paramters of the emitting systems