Gravitational wave phasing for spinning binaries

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Plan of the talk

- An attempt to obtain temporally evolving GW polarization states $h_{\times,+}(t)$ for spinning binaries inspiralling along eccentric orbits
- How to do GW phasing for such binaries
- Why did we argue that an approach by Cornish & Key is not quite appropriate for the above purpose

In collaboration with **Gerhard Schäfer**

Motivation

- It is desirable to develop a prescription that can provide gravitational wave (GW) templates for spinning compact binaries inspiralling along eccentric orbits
- A plausible application in the near future may be to model GWs from such binaries relevant for Pulsar Timing Array
- GWs from systems like OJ287 (modelled to contain a spinning SMBH binary) should be interesting from the nano-Hz GW window

GW phasing : I

- Aim is to obtain $h_{\times,+}(t)$ for spinning binaries inspiralling along eccentric orbits

- Consider $\times$ GW polarization state (Quadrupole in Amplitude)

$$h_\times\big|_{\mathrm{Q}} = \frac{4G\mu}{c^4 R'}\left[(\mathbf{p}\cdot\mathbf{v})(\mathbf{q}\cdot\mathbf{v}) - \frac{Gm}{r}(\mathbf{p}\cdot\mathbf{n})(\mathbf{q}\cdot\mathbf{n})\right] ,$$

- The observer triad $[\mathbf{p}, \mathbf{q}, \mathbf{N}]$ consists of the radial vector to the observer $\mathbf{N}$, $\mathbf{q} = \mathbf{N} \times \mathbf{p}$ & $\mathbf{p} = \mathbf{N} \times \mathbf{j}_0$

  $\mathbf{j}_0$: a unit vector along the total angular momentum at the initial epoch
  $\mathbf{r} = r\,\mathbf{n}$ & $\mathbf{v} = d\mathbf{r}/dt$

- Aim is to develop a prescription to obtain temporal evolution of these dot products via finding a way to obtain $\mathbf{r}(t)$ & $\mathbf{v}(t)$

GW phasing : II

- It is straightforward to write down $\mathbf{r}$ in the $(\mathbf{p}, \mathbf{q}, \mathbf{N})$ frame

$$\begin{aligned}\mathbf{r} \quad &= r\Bigg\{\Big[\cos\Phi\,\cos\alpha - \sin\Phi\cos\iota\,\sin\alpha\Big]\mathbf{p} + \Bigg[\Bigg(-\sin\iota S_\nu \\ &\quad + \cos\iota\cos\alpha\, C_\nu\Bigg)\sin\Phi + \cos\Phi\,\sin\alpha\, C_\nu\Bigg]\mathbf{q} + \Bigg[\Bigg(\sin\iota\, C_\nu \\ &\quad + \cos\iota\,\cos\alpha\, S_\nu\Bigg)\sin\Phi + \cos\Phi\sin\alpha\, S_\nu\Bigg]\mathbf{N}\Bigg\},\end{aligned}$$

  & a similar expression for $\mathbf{v}$
- Precession-al motion of $\mathbf{k}$, a unit vector along $\mathbf{L}$, will provide conservative $\alpha(t)$ & $\iota(t)$ as $\mathbf{k}$ in the invariant frame, defined w.r.t $\mathbf{j}_0$

$$\mathbf{k} = \sin\iota\,\sin\alpha\,\mathbf{e}_x - \sin\iota\,\cos\alpha\,\mathbf{e}_y + \cos\iota\,\mathbf{e}_z$$

  In practice, one solves precessional equations for $\mathbf{k}, \mathbf{s}_1$ and $\mathbf{s}_2$ together

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  In practice, one solves precessional equations for $\mathbf{k}, \mathbf{s}_1$ and $\mathbf{s}_2$ together
- We need ways to obtain conservative evolutions for $r(t)$ & $\Phi(t)$
  After which one may impose effects of gravitational RR on the underlying conservative dynamics

Conservative radial dynamics: I

- Take binaries containing two spinning compact objects $(m_1, m_2, \boldsymbol{\mathcal{S}}_1, \boldsymbol{\mathcal{S}}_2)$
  We want to obtain $r(t)$ while considering dynamics that include effects of dominant order spin-orbit interactions

$$H(\mathbf{r}, \mathbf{p}, \mathbf{S}_1, \mathbf{S}_2) = \frac{\mathbf{p}^2}{2} - \frac{1}{r} + \frac{(\mathbf{L} \cdot \mathbf{S}_{\text{eff}})}{c^2 r^3} \,,$$

  Scaled variables & effective spin
- It is easy to obtain an expression for $\dot{r} = p_r$

$$\mathrm{p}_r^2 = 2E + \frac{2}{r} - \frac{L^2}{r^2} - \frac{2(\mathbf{L} \cdot \mathbf{S}_{\text{eff}})}{c^2 r^3}$$

  We can show that $E, L, (\mathbf{L} \cdot \mathbf{S}_{\text{eff}})$ are conserved under the above Hamiltonian
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- Further, $\dot{r}^2$ also is a polynomial in $1/r$ with constant coefficients
  It is possible to obtain Keplerian type parametric solution

Conservative radial dynamics: II

- Radial part of the orbital dynamics is given by

$$r = a_r(1 - e_r \cos u)\,,$$
$$l \equiv n(t - t_0) = u - e_t \sin u\,,$$

where u and l are the eccentric and mean anomalies
The orbital elements can be explicitly given in terms of E, L and $\mathbf{L} \cdot \mathbf{S}_{\rm eff}$

- Our approach allows one to include (very easily) non-spinning PN corrections

$$r = a_r\left(1 - e_r \cos u\right)\,,$$
$$l \equiv n\left(t - t_0\right) = u - e_t \sin u + \frac{g_{4t}\left(v - u\right)}{c^4} + \frac{f_{4t}\,\sin v}{c^4}\,,$$

These expressions provide 2PN-accurate orbital dynamics (radial part) that include dominant order spin-orbit interactions

Conservative angular dynamics: I

- Angular part of our orbital dynamics is governed

$$r\dot{\theta} = \mathbf{e}_\theta \cdot \dot{\mathbf{r}} = \mathrm{p}_\theta + \frac{\mathbf{e}_\phi \cdot \mathbf{S}_{\mathrm{eff}}}{c^2 r^2} \,,$$

$$r \sin\theta \dot{\phi} = \mathbf{e}_\phi \cdot \dot{\mathbf{r}} = \mathrm{p}_\phi - \frac{\mathbf{e}_\theta \cdot \mathbf{S}_{\mathrm{eff}}}{c^2 r^2} \,,$$

  where θ is the angle between $\mathbf{n}$ and $\mathbf{j}_0$ & ϕ is the usual azimuthal angle [defined in the $\mathbf{e}_x - \mathbf{e}_y$ plane, while $\mathbf{j}_0$ is along $\mathbf{e}_z$]

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- The longitudinal & azimuthal motions are intertwined & it is rather impossible to tackle them analytically (without making further assumptions)
- In other words, it is difficult to find $\mathbf{r}(t)$ semi-analytically

$$\mathbf{r}(t) = r(t)\,\mathbf{n}(t)\,, \quad \text{where}$$

$$\mathbf{n}(t) = \sin\theta\cos\phi\,\mathbf{e}_x + \sin\theta\sin\phi\,\mathbf{e}_y + \cos\theta\mathbf{e}_z\,,$$

Conservative angular dynamics: II

- However, $h_{\times,+}$ depend on the usual 3 Eulerian angles (Φ, α, i) such that $\mathbf{r} = r\,\mathbf{n}$

$$\mathbf{n} = \Big\{ (\cos\Phi\,\cos\alpha - \sin\Phi\,\cos\iota\,\sin\alpha)\,\mathbf{e}_x + (\cos\Phi\,\sin\alpha + \sin\Phi\,\cos\iota\,\cos\alpha)\,\mathbf{e}_y + \sin\Phi\,\sin\iota\,\mathbf{e}_z \Big\},$$

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Recall that $r(t)$ is under control; while $i(t)$ & $\alpha(t)$ via $\dot{\mathbf{k}}, \dot{\mathbf{s}}_1, \dot{\mathbf{s}}_2$

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- The idea is to write $\mathbf{v} = \dot{\mathbf{r}}$ in a co-moving triad $(\mathbf{n}, \boldsymbol{\xi} = \mathbf{k} \times \mathbf{n}, \mathbf{k})$:

$$\mathbf{v} = \dot{r}\,\mathbf{n} + r\left(\frac{d\Phi}{dt} + \frac{d\alpha}{dt}\,\cos\iota\right)\boldsymbol{\xi} + r\left(\frac{d\iota}{dt}\,\sin\Phi - \sin\iota\,\cos\Phi\,\frac{d\alpha}{dt}\right)\mathbf{k}\,.$$

Conservative angular dynamics: III

- The expressions for $v^2 = \mathbf{v} \cdot \mathbf{v}$ arising from the above Equation & from our Hamiltonian allows us to obtain

$$\dot{\Phi} = \frac{L}{r^2} + \frac{(\mathbf{k} \cdot \mathbf{S}_{\text{eff}})}{c^2 r^3} - \cos i \, \dot{\alpha} \, ,$$

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- Recall that $\dot{\alpha}$ does not admit Keplerian type parametric solution
  This $\rightarrow \dot{\Phi}$ also will not have a semi-analytic solution (& hence requires numerical treatment)

- Our approach allows one to include non-spinning corrections to $\dot{\Phi}$ in an easy manner
  This allows us to obtain fully 2PN-accurate evolution for $r(t)$ and $\Phi(t)$

Including Reactive dynamics :I

- Gravitational RR is included in an adiabatic manner
  This is because RR time scale is longer than the time scales associated with the orbital and precessional motions.

- Therefore, the conserved quantities E and L appearing in the PN-accurate equations that govern $r(t), \Phi(t)$ and $\alpha(t)$ are allowed to vary with time

- It is natural to use far-zone energy & angular momentum fluxes to provide variations in E and L

- Further, it is also convenient to use e_t and $\omega = \dot{\Phi} + \cos\iota\,\dot{\alpha}$ while writing down expressions for the far-zone fluxes

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- This is our prescription to obtain $h_{\times,+}(t)$ from spinning compact binaries in inspiralling eccentric orbits

Including Reactive dynamics :II

- At present, we are working on a code that implements the above prescription to create easy-to-use $h_{\times,+}(t)$
- Developing a way to include spin-spin effects
- It will be desirable to probe data analysis implications

Including Reactive dynamics :II

- At present, we are working on a code that implements the above prescription to create easy-to-use $h_{\times,+}(t)$
- Developing a way to include spin-spin effects
- It will be desirable to probe data analysis implications
- This is because we initiated the project as Cornish & Key (2010) argued that angular part of the above dynamics is integrable

GW phasing CK's approach:I

- Cornish & Key (CK) argued that *Angular Part of the above Orbital Dynamics* admits a Keplerian type parametric solution in a a non-inertial frame that follows the precession of the orbital plane, defined by $\mathbf{L_N} = \mu\, \mathbf{r} \times \mathbf{v}$
- Recall that angular part of the orbital dynamics :

$$r\dot{\theta} = \mathbf{e}_\theta \cdot \dot{\mathbf{r}} = \mathrm{p}_\theta + \frac{\mathbf{e}_\phi \cdot \mathbf{S}_{\mathrm{eff}}}{c^2 r^2}\,,$$

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- Their precessing frame is given by

$$\left.\frac{d\mathbf{l_N}}{dt}\right|_{\mathrm{pre}} \equiv 0 = \frac{d\mathbf{l_N}}{dt} - \frac{1}{c^2 r^3}(\mathbf{S}_{\mathrm{eff}} \times \mathbf{l_N})\,,$$

where $\mathbf{l_N} = \mathbf{L_N}/|\mathbf{L_N}|$ & they introduced $\mathbf{v}_{\mathrm{pre}}$

GW phasing CK's approach:II

- Orbital velocity in the precessional frame is

$$\mathbf{v}_{\text{pre}} = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r}\,,$$

  where $\boldsymbol{\omega} = \mathbf{S}_{\text{eff}}/(c^2\, r^3)$ implying that the radial motion is unchanged

- CK argued that $\mathbf{L_N} \cdot \mathbf{v}_{\text{pre}} = 0$
  Orbital plane remains fixed in the precessing frame ..like in the Newtonian dynamics

- However, we showed that $\mathbf{L_N} \cdot \mathbf{v}_{\text{pre}} \neq 0$ at the the considered PN order

$$\mathbf{L_N} \cdot \mathbf{v}_{\text{pre}} = -\mu \left\{ \left(\mathbf{r} \times \mathbf{v} \right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r} \right) \right\} \neq 0$$

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- Further, invoking $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$ instead of $\mathbf{L_N}$ to describe the orbital plane will NOT help

- This simply means that $\mathrm{p}_\phi^2 + \mathrm{p}_\theta^2 = L^2/r^2$

GW phasing CK's approach:III

- How does one prove that angular part does NOT admit Keplerian type solution
- According to CK, angular part admits Keplerian-type solution $\mathbf{v}_{\rm pre} = d\mathbf{r}'/dt|_{\rm pre}$, where $\mathbf{r}'(t) = r(t)\left(\cos\varphi(t)\,\mathbf{p}'(t) + \sin\varphi(t)\,\mathbf{l}'(t)\right)$
  $\mathbf{p}' = \mathbf{k}\times\mathbf{N}'/|\mathbf{k}\times\mathbf{N}'|$ & $\mathbf{l}' = \mathbf{k}\times\mathbf{p}'$
  while $\mathbf{k}$ and $\mathbf{N}'$ are unit vectors along $\mathbf{L}$ and the line of sight vector to the source
- Let us compute from $\mathbf{r}'$, $\mathbf{v}^2_{\rm pre}$ and see we get the expected $v^2_{\rm prec} = \dot{r}^2 + r^2\dot{\varphi}^2$

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- Let us compute from $\mathbf{r}'$, $\mathbf{v}^2_{\rm pre}$ and see we get the expected $v^2_{\rm prec} = \dot{r}^2 + r^2\dot{\varphi}^2$
- $$v^2_{\rm prec} = \dot{r}^2 + r^2\dot{\varphi}^2 + r\,\dot{r}\left[(\dot{\mathbf{l}}'\cdot\mathbf{n}')\,\sin\varphi + (\dot{\mathbf{p}}'\cdot\mathbf{n}')\,\cos\varphi\right] + r^2\,\dot{\varphi}\,(\mathbf{l}'\cdot\dot{\mathbf{p}}')$$

  It is trivial to show that $\dot{\mathbf{p}}'\cdot\mathbf{n}' \neq 0$,
  Similar arguments hold for $\dot{\mathbf{l}}'\cdot\mathbf{n}'$ & $\mathbf{l}'\cdot\dot{\mathbf{p}}'$.

GW phasing CK's approach:IV

- Therefore, $\mathbf{r}' = r\left(\cos\varphi\,\mathbf{p}' + \sin\varphi\,\mathbf{l}'\right)$ will not give a $\mathbf{v}_{\rm pre}$ such that $v_{\rm prec}^2 = \dot{r}^2 + r^2\,\dot{\varphi}^2$
- This is true even while employing $\mathbf{L}$ to define the orbit.
- There are other issues associated with CK's approach..
- This is why we were forced to present our approach

Summary

- We discussed how to develop a way to create easy-to-use GW search templates for spinning compact binaries inspiralling along eccentric orbits
- It will be nice to probe its data analysis implications

THANK YOU !